COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

23,1 (1982)

FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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Abstract: We prove new fixed point theorems for multivalued mappings. Moreover, we construct a simple example which shows that the conjecture of J. P. Penot, stated in [8], is false.

Key words: Metric space, Banach space, fixed point theorems, multivalued mappings.

Classification: Primary 47H10, 47H15
Secondary 54C60

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1. A fixed point theorem for multivalued mappings in complete metric spaces.

Let M be a metric space with metric d, A, B being subsets of M, $x_0 \in M$. Put: $d(x_0, A) = \inf\{d(x_0, x) : x \in A\}$, $D(A,B) = \{\lambda > 0 : A \subseteq V_\lambda(B)$ and $B \subseteq V_\lambda(A)\} = \max\{\sup\{d(x,B) : x \in A\}, \sup\{d(y,A) : y \in B\}\}$, where $V_\lambda(A) = \{y \in M, d(y,Y) \le \lambda\}$ for $\lambda > 0$.

Definition 1. Let M be a metric space with metric d. We say that a map $f: M \to M$ satisfies the Caristi's condition if there exists a lower semicontinuous function $h: M \to R_+ = [0, \infty)$ such that $d(x, f)) \le h(x) - h(f(x))$ for all $x \in M$.

Theorem 1. Let M be a complete metric space, $F: M \to M$ be a multivalued mapping of M into the family of all nonempty compact subsets of M such that $D(F(x), F(y)) < d(x,y)$ for all

$x \neq y \in M$. Suppose that there exists a single-valued map $f:$ $: M \longrightarrow M$ satisfying the Caristi's condition such that:

1) $d(x,F(x)) \geq \inf \{ d(f^n(x)), F(f^n(x)) : n=1,2,\dots \}$ for all $x \in M$, where $f^n(x) = (\underbrace{f \circ f \circ \dots \circ f}_{n\text{-times}})(x)$,

2) $K = \{ x \in M,\ f(x) = x \}$ is precompact.

Then F has a fixed point in M.

Proof. We claim that for each $z \in M$ there exists a $z_0 \in K$ such that $d(z_0,F(z_0)) \leq d(z,F(z))$. Let $h: M \longrightarrow R_+$ be a lower semicontinuous function such that $d(x,f(x)) \leq h(x) - h(f(x))$ for all $x \in M$. We write $x \leq y$ iff $d(x,y) \leq h(x) - h(y)$. Then $\leq$ is a partial order on M. Let z be an arbitrary fixed point in M. Put $M_z = \{ x \in M : d(x,F(x) \leq d(z,F(z)) \}$. Then M_z is a non-empty ($z \in M_z$) closed subset of M, since $d(x,F(x))$ is a continuous function on M. Therefore M_z is complete. Using the same argument as in [8] one can prove that there exists a maximal element z_0 in M_z (i.e. if $x \in M_z$ and $x \geq z_0$ then $x = z_0$).

Suppose that there exists an $n \in N$ such that $d(f^n(z_0),\ F(f^n(z_0))) \leq d(z_0,F(z_0)) \leq d(z,F(z))$

Then $f^n(z_0) \in M_z$. On the other hand, we have:

$d(z_0,f(z_0)) \leq h(z_0) - h(f(z_0))$, $d(f(z_0),\ f^2(z_0)) \leq h(f(z_0)) - h(f^2(z_0))$ $\dots$, $d(f^{n-1}(z_0),\ f^n(z_0), \leq h(f^{n-1}(z_0)) - h(f^n(z_0))$.

Hence

$$d(z_0,f^n(z_0)) \leq \sum_{i=1}^{n} d(f^{i-1}(z_0),f^i(z_0)) \leq h(z_0) - h(f^n(z_0)),$$

where $f^0(z_0) = z_0$. This implies $f^n(z_0) \geq z_0$, $f^n(z_0) \in M_z$. Hence $f^n(z_0) = z_0$ and it is clear that $f(z_0) = z_0 \in K \cap M_z$.

Now suppose that $d(f^n(z_0),F(f^n(z_0)) > d(z_0,F(z_0))$ for all n. Then there exists a subsequence $\{n_i\}$ such that $\lim_i d(f^{n_i}(z_0),\ F(f^{n_i}(z_0))) = d(z_0,F(z_0))$. It is easy to see that $\{f^n(z_0)\}$ is a Cauchy sequence in M. Then there exists a point $z_\infty \in M$ such that $z_\infty = \lim f^{n_i}(z_0)$, since M is complete. Hence

$$d(z_0,z_\infty) = \lim d(z_0,f^{n_i}(z_0)) \leq h(z_0) - \lim h(f^{n_i}(z_0)) \leq$$
$$\leq h(z_0) - h(z_\infty),$$
$$d(z_\infty,F(z_\infty)) = \lim_i d(f^{n_i}(z_0), F(f^{n_i}(z_0))) = d(z_0,F(z_0)) =$$
$$\leq d(z,F(z)).$$

This means that $z_\infty \in M_z$ and $z_\infty \geq z_0$. Therefore $z_\infty = z_0$ and $h(z_\infty) = h(f(z_0)) = h(z_0)$. Hence $d(f(z_0),F(f(z_0))) = d(z_0,F(z_0))$. This contradicts the assumption $d(f^n(z_0),F(f^n(z_0))) > d(z_0,F(z_0))$ for all n=1,2,... . This proves our claim.

It is easy to see that $\inf\{d(x,F(x)):x \in M\} = \inf\{d(x,F(x)):x \in \overline{K}\}$. Since $\overline{K}$ is compact, there exists a point $x_0 \in \overline{K}$ such that $d(x_0,F(x_0)) = \inf\{d(x,F(x)):x \in M\}$. If $r = d(x_0,F(x_0)) > 0$, take a $y \in F(x_0)$ such that $d(x_0,y) = d(x_0,F(x_0)) = r$. Then $d(y,F(y)) \leq D(F(x_0),F(y)) < d(x_0,y) = r$. This contradicts the assumption $d(x_0,F(x_0)) = \inf\{d(x,F(x)):x \in M\}$. Hence $d(x_0,F(x_0)) = 0$ and $x_0 \in F(x_0)$. This completes the proof.

Remark: In [8] J.P. Penot has stated the following problem: Let M be a complete metric space, $h:M \to R_+$ be a lower semicontinuous function and $F:M \to M$ be a multivalued mapping of M into the family of all nonempty closed subsets of M satisfying the following condition:
$d(x,F(x)) \leq h(x) - \inf\{h(y):y \in F(x)\}$. Does F have a fixed point in M ?

The following simple example shows that this conjecture is false.

Put $M = [0,\infty)$ with the usual metric. Put $h(x) = \frac{1}{1+x}$ $F(x) = [x + \frac{1}{2(1+x)}, 2x+1]$ for all $x \in M$. Then M is a complete metric space, $h:M \longrightarrow R_+$ is continuous, F satisfies the condition $d(x,F(x)) \quad h(x) - \inf\{h(y): y \in F(x)\}$, but F has not any fixed point in M.

<u>Proposition I</u>. Let M be a complete metric space, $h: :M \longrightarrow R_+$ be a lower semicontinuous function, $F:M \longrightarrow M$ be a multivalued mapping which maps M into the family of all nonempty closed subsets of M. Suppose F satisfies the following condition $\inf\{d(x,y) + h(y): y \in F(x)\} \leq h(x)$ for all $x \in M$. Then F has a fixed point in M.

<u>Proof</u>. We claim that for each $x \in M$ there exists an $f(x) \in$ $\in F(x)$ such that $d(x,f(x)) \leq 2\,h(x) - 2\,h(f(x))$. If $d(x,F(x)) =$ $= 0$, put $f(x) = x$. If $d(x,F(x)) > 0$, then

$d(x,F(x)) + \inf\{d(x,y) + 2h(y): y \in F(x)\} \leq 2\inf\{d(x,y) +$ $+ h(y): y \in F(x)\} \leq 2h(x)$.

It follows that $\inf\{d(x,y) + 2h(y): y \in F(x)\} < 2h(x)$. Then there exists a point $f(x) \in F(x)$ such that $d(x,f(x)) + 2h(f(x)) \leq$ $\leq 2h(x)$. This proves our claim.

According to Christi's Theorem there exists a point $x_0 \in M$ such that $x_0 = f(x_0) \in F(x_0)$. This completes the proof.

<u>Corollary 1</u>(S.B. Nadler [7]). Let M be a complete metric space. If $F:M \longrightarrow M$ is a multivalued contraction mapping which maps M into the family of all nonempty closed subsets of M, then F has a fixed point.

<u>Proof</u>. Let $D(F(x),F(y)) \leq kd(x,y)$, where $0 \leq k < 1$. Put

$h(x) = \frac{1}{1-k} d(x,F(x))$. Then

$\inf\{d(x,y) + h(y): y \in F(x)\} = \inf\{d(x,y) + \frac{1}{1-k} d(y,F(y)):$
$: y \in F(x)\} = \inf\{d(x,y) + \frac{1}{1-k} \cdot D(F(x),F(y)): y \in F(x)\} \leq$
$\leq \inf\{d(x,y) + \frac{1}{1-k} k\, d(x,y): y \in F(x)\} = \frac{1}{1-k} d(x,F(x)) = h(x).$
By Proposition 1, F has a fixed point in M.

Corollary 2. Let M, h, F be as in Proposition 1.

1. If $d(x,F(x)) \leq h(x) - \sup\{h(y): y \in F(x)\}$, then F has a fixed point in M.

2. If $D(\{x\},F(x)) \leq h(x) - \inf\{h(y): y \in F(x)\}$, then there exists an $x_0 \in M$ such that $f(x_0) = \{x_0\}$.

Proof. It is clear that F has a fixed point in M, because $\inf\{d(x,y) + h(y): y \in F(x)\} \leq d(x,F(x)) + \sup h(F(x))$ and $\inf\{d(x,y) + h(y): y \in F(x)\} \leq D(\{x\},F(x)) + \inf\{h(y): y \in F(x)\}$. To prove 2, it is sufficient to note that for each $x \in M$ there exists a point $f(x) \in F(x)$ such that
$D(\{x\},F(x)) \leq h(x) - \inf\{h(y): y \in F(x)\} \leq 2h(x) - 2h(f(x))$.
By Caristi's Theorem there exists a point $x_0 \in M$ such that $x_0 = f(x_0)$. Then $D(\{x_0\},F(x_0)) \leq 2h(x_0) - 2h(f(x_0)) = 0$. It follows that $F(x_0) = \{x_0\}$. This completes the proof.

2. A fixed point theorem for multivalued mappings in Banach spaces

Definition 2. Let X, Y be topological spaces, $F: X \to Y$ be a multivalued mapping. We say that F is upper semicontinuous at $x \in X$ if for each open set $G \subset Y$, $F(x) \subset G$ there exists a neighborhood U of x such that for each $x' \in U$ we have $F(x') \subseteq G$.

Theorem 2. Let X be a Banach space, $C \subseteq X$ be a convex closed nonempty bounded subset of X, $F: C \longrightarrow C$ be a multivalued nonexpansive mapping which maps into the family of all nonempty convex closed subsets of C. Suppose that there exist a function $\mu: R_+ \longrightarrow R_+$ which is nondecreasing and $\mu(t) > 0$ for all $t > 0$, a function $\varphi: C-C \longrightarrow R$ weakly continuous at θ, $\varphi(\theta) > 0$ and a mapping $\psi: C-C \rightarrow \mathscr{C}(X^*)$, where $\mathscr{C}(X^*)$ denotes the family of all nonempty closed subsets of the dual space X^*, weakly-strongly upper-semicontinuous at θ, $\psi(\theta)$ is compact, such that

$$d(x,F(x)) + d(y,F(y)) \geq \mu(\| x-y \|)\, \varphi(x-y) - \psi_s(x-y)$$

for all $x, y \in C$, where $\psi_s(x) = \sup \{ |\langle x^*, x \rangle| \mid x^* \in \psi(x) \}$. Then F has a fixed point in C.

Proof. By the boundness of C, there exists a number $M > 0$ such that $C \subseteq B_M = \{ x \in X : \| x \| \leq M \}$. Hence $C - C \subseteq B_{2M}$. By the standard argument there exists a sequence $\{x_n\} \subseteq C$ such that $d(x_n, F(x_n)) < \frac{1}{n}$ for each $n \in N$. Since $\{x_n\}$ is bounded in X, $\{x_n\}$ is weakly precompact. Then there exists a weakly Cauchy subnet $\{x_{\wp(i)}\}_{i \in I}$ of $\{x_n\}$ where $\wp: I \longrightarrow N$. Then it is clear that the net $\{u_{i,j}\}_{(i,j) \in I \times I}$ where $u_{i,l} = x_{\wp(i)} - x_{\wp(j)}$ converges weakly to θ.

We claim that $\lim \| u_{i,j} \| = 0$. Suppose that it is false. There exists a number $r > 0$ such that for any $(i,j) \in I \times I$ there exists an $(i',j') \in I \times I$, $(i',j') \geq (i,j)$ and $\| u_{i,j} \| \geq r$. Since φ is weakly continuous at θ we have $\lim \varphi(u_{i,j}) = \varphi(\theta) = k > 0$. Let $\tau: X \longrightarrow X^{**}$ be a canonical embedding map of X into its bidual space X^{**}. Since $\{\tau(u_{i,j})\}$ is bounded in X^{**}, $\{\tau(u_{i,j})\}$ is an equicontinuous family of mappings from $(X^*, \|\cdot\|)$ into R. Since $\{\tau(u_{i,j})\}$ converges pointwise to Θ on X^* and $\psi(\Theta)$ is a compact subset of X^* by Theorem 4.5 [9, chapt. III] it follows that $\{\tau(u_{i,j})\}$ converges uniformly to Θ on $\psi(\Theta)$. Then there exists an index $(i_0, j_0) \in I \times I$ such that for $(i,j) \in I \times I$, $(i,j) \geq (i_0, j_0)$ we get $\varphi(u_{i,j}) \geq \frac{3}{4} \cdot k$ and $|\langle \tau(u_{i,j}), x^* \rangle| = |\langle x^*, u_{i,j} \rangle| \leq \frac{1}{8} k\mu(r)$ for all $x^* \in \psi(\Theta)$. Since ψ is weakly-strongly upper-semicontinuous at Θ and $\{u_{i,j}\}$ converges weakly to Θ, there exists an index $(i_1, j_1) \in I \times I$, $(i_1, j_1) \geq (i_0, j_0)$ such that:

$\psi(u_{i,j}) \subseteq \psi(\Theta) + \frac{k\mu(r)}{16M} B_1^*(\Theta)$, where $B_1^* = \{x^* \in X^* : \|x^*\| \leq 1\}$

for all $(i,j) \in I \times I$, $(i,j) \geq (i_1, j_1)$. Then

$$\psi_s(u_{i,j}) = \sup \{|\langle x^*, u_{i,j}\rangle| : x^* \in \psi(u_{i,j})\} \leq$$
$$\leq \sup\{|\langle x^*, u_{i,j}\rangle| : x^* \in \psi(\Theta) + \frac{k\mu(r)}{16M} B_1^*(\Theta)\} \leq$$
$$\leq \sup\{|\langle x^*, u_{i,j}\rangle| : x^* \in \psi(\Theta)\} + \frac{k\mu(r)}{16M} \sup\{|\langle x^*, u_{i,j}\rangle| :$$
$$: x^* \in B_1^*\} \leq \frac{k\mu(r)}{8} + \frac{k\mu(r)}{16M} \|u_{i,j}\| \leq \frac{k\mu(r)}{4}$$

for all $(i,j) \in I \times I$, $(i,j) \geq (i_1 j_1)$.

Take $n, m \in N$ such that $\frac{1}{n} + \frac{1}{m} < \frac{k\mu(r)}{2}$. Choose $i_2 \in I$, $i_2 \geq i_1$, $i_2 \geq j_1$ such that $\wp(i) \geq \max\{n, m\}$ for all $i \in I$, $i \geq i_2$. Take $(i_3, j_3) \in I \times I$, $(i_3, j_3) \geq (i_2, i_2)$ such that $\|u_{i_3, j_3}\| \geq r$. Then

$$d(x_{\wp(i_3)}, F(x_{\wp(i_3)})) + d(x_{\wp(j_3)}, F(x_{\wp(j_3)})) \geq$$
$$\geq \varphi(u_{i_3, j_3})\mu(\|u_{i_3, j_3}\|) - \psi_s(u_{i_3, j_3}).$$

Hence

$$\frac{1}{n} + \frac{1}{m} \geq \frac{1}{\wp(i_3)} + \frac{1}{\wp(j_3)} \geq d(x_{\wp(i_3)}, F(x_{\wp(i_3)})) +$$
$$+ d(x_{\wp(j_3)}, F(x_{\wp(j_3)})) \geq \frac{3}{4} k\mu(r) - \frac{k\mu(r)}{4} = \frac{1}{2} k\mu(r).$$

This contradicts $\frac{1}{n} + \frac{1}{m} < \frac{1}{2} k\mu(r)$ and this proves our claim.

H is analytic, then F is lower semicontinuous provided it has a Souslin graph (Theorem 7). This version of the Souslin-graph theorem is based on ideas due to Frolík [7] and [6].

2. <u>Almost lower semicontinuous multifunctions</u>. Almost continuous mappings were considered first, as it seems, by Blumberg [3] and Block and Cargal [2], under unlike names. The term "almost continuity" was used by Bradford and Goffman [4].

Let X and Y be topological spaces and f a mapping of X to Y ($f:X \longrightarrow Y$). Given $x \in X$, f is said to be almost continuous at x if for each open set V in Y containing $f(x)$, $x \in \text{Int } D(f^{-1}(V))$. Here $D(E)$, where $E \subset X$, denotes (as in [10]) the set of all points x' of X that are of second category in X relative to E (i.e. $U \cap E$ is of second category in X for each open $U \ni x'$). This definition of almost continuity is equivalent to those given in the above-mentioned papers, and can be extended, in a natural way, to multifunctions. By a multifunction F of X to Y ($F:X \longrightarrow Y$) we mean a function which to every point $x \in X$ assigns a subset $F(x)$ of Y (not necessarily closed or nonempty).

<u>Definition</u>. A multifunction $F:X \longrightarrow Y$ is almost lower semicontinuous at x if for every open set V in Y

$$x \in F^{-1}(V) \text{ implies } x \in \text{Int } D(F^{-1}(V)).$$

Here the inverse image $F^{-1}(V)$ denotes, as always, the set of all x' satisfying $F(x') \cap V \neq \emptyset$.

The set of all points x of X such that F is almost lower semicontinuous at x will be denoted by $L_a(F)$; in case $L_a(F) = X$, F will be called almost lower semicontinuous. Thus, F is almost lower semicontinuous if and only if for every open set V in Y

$$F^{-1}(V) \subset \text{Int } D(F^{-1}(V)).$$

Let $L(F)$ stand for the set of all points $x \in X$ such that F is lower semicontinuous at x, i.e. $x \in \text{Int } F^{-1}(V)$ for all open $V \subset Y$ intersecting $F(x)$. Notice that

$$L_a(F) \cap F^{-1}(Y) \subset \text{Int } D(F^{-1}(Y)) \subset \text{Int } D(X)$$

and

$$L(F) \cap F^{-1}(Y) \cap \text{Int } D(X) \subset L_a(F),$$

while obviously $X \setminus F^{-1}(Y) \subset L(F) \cap L_a(F)$. In particular, if X is a Baire space (i.e. $X = D(X)$), then $L(F) \subset L_a(F)$. If F is almost lower semicontinuous, then $F^{-1}(Y)$ is a Baire space (in itself).

The usefulness of the property of almost lower semicontinuity stems from the fact that it is automatically satisfied under some category-type assumptions, while, on the other hand, it is a convenient starting point to the Souslin-graph, closed graph, open mapping and Blumberg theorems.

The following theorem extends some observations from [3], [2] and [4].

Theorem 1. Let F be a multifunction of X to Y. If the space Y is second-countable, then

(i) The set $L_a(F)$ is residual in X;

(ii) the restriction $F|L_a(F) : L_a(F) \longrightarrow Y$ is almost lower semicontinuous. More generally, for each residual set $A \subset L_a(F)$, $F|A$ is almost lower semicontinuous.

Proof. (i) Let $\{V_n\}$ be a base for Y. A point $x \in X$ is not in $L_a(F)$ if and only if there is n such that $x \in F^{-1}(V_n)$

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

23,1 (1982)

ALMOST LOWER SEMICONTINUOUS MULTIFUNCTIONS AND THE SOUSLIN-GRAPH THEOREM

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Abstract: Almost continuous mappings and almost lower semicontinuous multifunctions are investigated. A Souslin-graph theorem for multihomomorphisms with values in an analytic space is proved.

Key words: Multifunction, almost lower semicontinuity, Souslin-graph.

Classification: 54C60

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1. Introduction. The term "almost continuity" is used here in the sense of Bradford and Goffman [4]. We show that each almost continuous mapping having the Baire property and taking values in a regular space is continuous (Theorem 4). It follows that each almost continuous mapping having a Souslin graph and taking values in an analytic space is continuous (Theorem 6).

We define and investigate "almost lower semicontinuity" of multifunctions. Under category type assumptions certain multifunctions possess automatically this property (Theorems 1,2,3).

Let $F:G \to H$ be a multihomomorphism with $F^{-1}(H) = G$. If G is of second category and H is separable or Lindelöf, then F is lower semicontinuous iff it is lower-Baire (Theorem 5). If G is inductively generated by second category groups and

Since $\lim \| u_{i,j} \| = 0$, it follows that $\{x_{\varphi(i)}\}$ is a Cauchy net in the strong topology. Therefore $\{x_{\varphi(i)}\}$ converges strongly to an $x \in C$. Then for $i \in I$, we have

$$d(x,F(x)) \leq \| x - x_{\varphi(i)} \| + d(x_{\varphi(i)},F(x_{\varphi(i)})) +$$
$$+ D(F(x_{\varphi(i)}),F(x)) \leq 2 \| x - x_{\varphi(i)} \| + \frac{1}{\varphi(i)}.$$

Hence $d(x,F(x)) = 0$. It follows that $x \in F(x)$ and this completes the proof.

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(Oblatum 1.4. 1981)

and $x \notin \operatorname{Int} D(F^{-1}(V_n))$. Thus

$$L_a(F) = X \setminus \bigcup_{n=1}^{\infty} [F^{-1}(V_n) \setminus \operatorname{Int} D(F^{-1}(V_n))].$$

Each set of the form $E \setminus \operatorname{Int} D(E)$ is of first category because $D(E) \setminus \operatorname{Int} D(E)$ is closed co-dense and $E \setminus D(E)$ is of first category by the Banach category theorem (cf. [10]). Hence $L_a(F)$ is residual.

(ii) Let $A \subset L_a(F)$ be residual in X. Put $E = F^{-1}(V_n)$. Then

$$A \cap D(E) = A \cap D(A \cap E) \subset D_A(A \cap E)$$

and

$$x \in A \cap \operatorname{Int} D(E) \subset \operatorname{Int}_A(A \cap D(E)) \subset \operatorname{Int}_A D_A(A \cap E) \text{ for } x \in A \cap E,$$

which shows almost lower semicontinuity of $F|A$.

By a graph of a multifunction $F:X \longrightarrow Y$ we mean the set

$$\operatorname{Gr} F = \{(x,y) : y \in F(x)\} \subset X \times Y.$$

In the following the letters G, H stand for topological groups. We say that $F:G \to H$ is a multihomomorphism if Gr F is a subgroup of $G \times H$. For multihomomorphisms we have the following simple criterion of almost lower semicontinuity.

Lemma 1. A multihomomorphism $F:G \longrightarrow H$ is almost lower semicontinuous if (and only if) for each neighbourhood V of e_H the inverse image $F^{-1}(V)$ is of second category in G.

Proof. Let V be a symmetric neighbourhood of e_H and put $E = F^{-1}(V)$ and $U = \operatorname{Int} D(E)$. Since $E \setminus U$ is a first category set in G (by the Banach category theorem; see the previous proof) and E is of second category (by the hypothesis), the set U is non-empty. This implies that $e_G \in \operatorname{Int} D(F^{-1}(V^2))$ because

$e_G \in U^2 \subset \mathrm{Int}\,([D(F^{-1}(V))]^2) \subset \mathrm{Int}\, D([F^{-1}(V)]^2) = \mathrm{Int}\, D[F^{-1}(V^2)]$.

Thus $e_G \in L_a(F)$. If now $x \in F^{-1}(V)$, where V is open in H, then $e_G \in F^{-1}(Vy^{-1})$, where $y \in F(x) \cap V$; hence $e_G \in \mathrm{Int}\, D[F^{-1}(Vy^{-1})]$, and so $x \in \mathrm{Int}\, D([F^{-1}(Vy^{-1})]x) = \mathrm{Int}\, D[F^{-1}(V)]$; $x \in L_a(F)$.

Now we need a generalization of a lemma of Pettis [13] for multihomomorphisms. First a definition ([13]). A subset E of H is σ-bounded in H if for every neighbourhood V of e_H there exists a sequence $\{y_n\} \subset E$ such that $E \subset \bigcup_{n=1}^{\infty} y_n V \cup V y_n$. Each separable or Lindelöf (in particular, σ-compact) subspace E of H is σ-bounded in H. If H is metrizable, the three notions (σ-boundedness, separability and Lindelöf property of $E \subset H$) coincide.

Lemma 2. Let $F: G \to H$ be a multihomomorphism such that $F(G)$ is σ-bounded in H. If $F^{-1}(H)$ is a second category set in G, then so is $F^{-1}(V)$ for any neighbourhood V of e_H.

Proof. Given open $V \ni e_H$, choose $\{y_n\} \subset F(G)$ so that $F(G) \subset \bigcup_{n=1}^{\infty} y_n V \cup V y_n$. Choose $x_n \in F^{-1}(y_n)$. Then $F^{-1}(H) = \bigcup_{n=1}^{\infty} x_n E \cup E x_n$, where $E = F^{-1}(V)$. Hence E is of second category in G.

Lemma 3 ([13]). If H is σ-bounded, then each set $E \subset H$ is σ-bounded in H.

Proof. Let $H = \bigcup_{n=1}^{\infty} y_n V \cup V y_n$, where V is a neighbourhood of e_H and $\{y_n\} \subset H$. Choose $h_n^1 \in E \cap y_n V$ whenever possible ($n \in N_1$) and $h_n^2 \in E \cap V y_n$ whenever possible ($n \in N_2$). Then

$$E \subset \bigcup_{n \in N_1} y_n V \cup \bigcup_{n \in N_2} V y_n \subset \bigcup_{n \in N_1} h_n^1 V^{-1} V \cup \bigcup_{n \in N_2} V V^{-1} h_n^2.$$

Since $V \ni e_H$ was arbitrary, E is σ-bounded.

If now H is σ-bounded, then $F(G)$ is σ-bounded in H

(Lemma 3) and the lemmas Nos. 2 and 1 may be applied, provided $F^{-1}(H)$ is of second category. Thus we get

Theorem 2. Let F be a multihomomorphism of G to H, where H is a σ-bounded group (e.g. separable or Lindelöf). If $F^{-1}(H)$ is of second category in G, then F is almost lower semicontinuous.

For linear multifunctions the assumption of σ-boundedness of the range space may be omitted and the proof reduced. Let S and T be topological vector spaces; $F:S \longrightarrow T$ is a linear multifunction if Gr F is a linear subspace of $S \times T$.

Theorem 3. Each linear multifunction $F:S \longrightarrow T$ such that $F^{-1}(T)$ is of second category in S is almost lower semicontinuous.

Proof. Let V be a neighbourhood of 0_T. Since $T = \bigcup_{n=1}^{\infty} nV$, $F^{-1}(T) = \bigcup_{n=1}^{\infty} nF^{-1}(V)$. Hence $F^{-1}(V)$ is of second category in S and we apply Lemma 1.

That is all about "automatic" almost lower semicontinuity. Now we will consider the question, when almost lower semicontinuity (resp. almost continuity) implies lower semicontinuity (resp. continuity). For mappings we have a quite satisfactory answer:

Theorem 4. Let X be a Baire space, and let Y be a regular space (even not necessarily T_0). A mapping $f:X \longrightarrow Y$ is continuous if (and only if) it is almost continuous and has the Baire property.

Proof. Let $x \in f^{-1}(V)$, where V is open in Y. Choose open set $W \subset Y$ with $f(x) \in W$ and $\overline{W} \subset V$. Since f is almost continuous at x, $x \in U = \text{Int } D(f^{-1}(W))$. Let $u \in U$; we will show that $f(u) \in \overline{V}$. Let Z be an open neighbourhood of $f(u)$. Since f is almost continuous at u, $u \in \text{Int } D(f^{-1}(Z))$. Hence $U \cap f^{-1}(Z)$ is a second category set in X. Since f has the Baire property, there exists an open set $G \subset X$ such that $G \Delta f^{-1}(Z)$ is of first category in X. Now $U \cap G$ is of second category in X. It follows that $f^{-1}(W) \cap G$ is of second category in X. Hence $f^{-1}(W) \cap f^{-1}(Z)$ is of second category in X, which yields $W \cap Z \neq \emptyset$. Thus we have proved that $f(u) \in \overline{W}$.

The theorem cannot be extended to multifunctions, without additional assumptions.

Example 1. Each of the following multifunctions is almost lower semicontinuous and lower-Baire (i.e. $F^{-1}(V)$ has the Baire property whenever $V \subset Y$ is open), but not lower semicontinuous.

(a) $F(x) = \{1\}$ for $x \in I \setminus Q$ and $F(x) = \emptyset$ for $x \in I \cap Q$ ($I = [0,1]$, Q - the rationals); $F: I \longrightarrow I$ is single-valued.

(b) $F(x) = Y$ for $x \in I \setminus Q$ and $F(x) = \{1\}$ for $x \in I \cap Q$, where Y is the discrete space $\{0,1\}$; $F^{-1}(Y) = I$.

(c) (cf. [5]). Let $I = P_1 \cup P_2$, where P_i are dense and co-dense G_δ-sets in I, and let g be the natural mapping of the space $Y = P_1 \oplus P_2$ onto I; g is continuous and almost open (i.e. $g(U) \subset \text{Int } D(g(U))$ for each open $U \subset Y$). Define $F = g^{-1}$.

An analogue of Theorem 4 for multihomomorphisms holds true. To see this, let $F: G \to H$ be an almost lower semicontinuous lower-Baire multihomomorphism, and consider the induced mapping $f: X \dashrightarrow Y$, where $X = F^{-1}(H)$ and $Y = H/F(e_G)$ (Y need not be a T_0-space of a group). The assumptions of Theorem 4 are satisfied (Y is regular by [9; 5.19, 5.20]). Hence f is continuous. Since the quotient mapping $\varphi: H \longrightarrow Y$ is open (cf. [9; 5.17]), this implies lower semicontinuity of $F: X \longrightarrow H$. X is a second category subgroup of G having the Baire property; by the Banach-Kuratowski-Pettis theorem (cf. [1; Theorem 1], [10; 13.XI] and [13; Theorem 1]), X is open in G. Hence $F: G \longrightarrow H$ is lower semicontinuous. Thus, in view of Theorem 2, we get

Theorem 5. Let F be a multihomomorphism of G to H such that $F^{-1}(H)$ is of second category in G. (i) F is lower semicontinuous if (and only if) it is almost lower semicontinuous and lower-Baire. (ii) Suppose the group H is σ-bounded (e.g. separable or Lindelöf). Then F is lower semicontinuous if (and only if) it is lower-Baire.

For linear F, (ii) holds with no assumption on the range vector space (by Theorem 3).

3. Souslin-graph theorem. A T_3-space Y is said to be an analytic space (or a K-Souslin space) if there exists a Polish space X and a compact-valued upper semicontinuous multifunction Φ of X onto Y (Frolík [6]; for some equivalent definitions see [6] and [8]). Each analytic space is a Lindelöf space, hence paracompact and normal (cf. [6] and [8]).
By a Souslin set, in a given space, we mean the result of performing the Souslin operation (A) (denoted also S) on a system of closed sets in the space. Since the collection of all sets having the Baire property is closed under the operation

(A), each Souslin set has the Baire property (cf. [10]).

L. Schwartz [15] proved that if S and T are locally convex spaces, S - ultrabornological (i.e. inductive limit of Banach spaces), T - continuous image of a Polish space, then each borel graph linear map $f:S \longrightarrow T$ is continuous and each continuous linear map $g:T \xrightarrow{onto} S$ is open.

Frolík [7] proved that if G is a vector space which is inductively generated by second category vector spaces and H is an analytic locally convex space, then

(1) each Souslin-graph homomorphism $f:G \longrightarrow H$ is continuous.

Martineau [11] proved, among other results, that if G is a second category analytic group and H is an analytic group, then each continuous homomorphism $g:H \xrightarrow{onto} G$ is open; Perez Carreras [12] showed that the theorem remains true if G is not necessarily analytic.

In this section we shall show that if G is inductively generated by second category groups and H is an analytic group, then the statements (1) and (2) hold, where

(2) each Souslin-graph homomorphism $g:H \xrightarrow{onto} G$ is open.

The main tools are Theorem 5 and the following lemma due to Rogers and Willmott [14] (a nice proof is given in Frolík [7; Lemma 1]).

<u>Lemma 4</u>. Let $F:X \longrightarrow Y$ be a multifunction, where Y is an analytic space. If Gr F is a Souslin set in $X \times Y$, then F is upper-Souslin (i.e. $F^{-1}(A)$ is Souslin whenever A is closed), and hence upper-Baire.

Combining the lemma with Theorem 4 we get

Theorem 6. Let X be a Baire space, and let Y be an analytic space. A mapping $f:X \longrightarrow Y$ is continuous if (and only if) it is almost continuous and its graph is a Souslin set in $X \times Y$.

Example 1 (c) shows that an almost lower semicontinuous Souslin-graph multifunction $F:X \longrightarrow Y$ need not be lower semicontinuous, even if X is compact, Y Polish and Gr F closed.

Lemma 5. Each upper Baire multihomomorphism $F:G \longrightarrow H$ is lower-Baire.

Proof. Let φ be the canonical mapping of H onto $H/F(e_G)$. Let V be open in H. Since φ is open and continuous, the set

$$F^{-}(V) = F^{-1}(H) \setminus F^{-1}(\varphi^{-1}[H/F(e_G) \setminus \varphi(V)])$$

has the Baire property in G.

If F is lower-Baire and $F(e_G)$ is compact, then φ is closed (cf. [9; 5.18]) and, consequently, F is upper-Baire. Without the compactness assumption, the converse to Lemma 5 is not true.

Example 2. Let H_0 be a closed normal subgroup of H, $G = H/H_0$, $\varphi:H \longrightarrow G$ the canonical quotient mapping and $F = \varphi^{-1}: G \longrightarrow H$; F is even lower semicontinuous and has a closed graph. Nevertheless F need not be upper-Baire (it is upper-Baire provided H is analytic; see Lemma 4). Take for instance $H = R \times R_d$ and $H_0 = \{0\} \times R_d$, where R_d denotes R (the reals) endowed with the discrete topology. Choose a set A in R which has not the Baire property and put $K = \{(x,x) \in H : x \in A\}$; K is closed and $F^{-1}(K) = A$.

Now we are in a position to derive the Souslin-graph theorem.

Theorem 7. Let $F:G \longrightarrow H$ be a multihomomorphism, where H is an analytic group. Assume that

(i) $F^{-1}(H)$ is of second category in G; or

(ii) $F^{-1}(H) = G$ and the topology on G is inductively generated by homomorphisms $h_\alpha : G_\alpha \longrightarrow G$, where $\{G_\alpha : \alpha \in A\}$ is a family of second category groups.

If Gr F is a Souslin set in $G \times H$, then F is lower semicontinuous.

Proof. (i) Follows from Lemmas 4, 5 and Theorem 5 (ii).

(ii) Fix any $\alpha \in A$. By Lemma 4, F is upper-Souslin; hence $F \circ h_\alpha$ is upper-Souslin, and so upper-Baire. Lemma 5 shows that $F \circ h_\alpha$ is lower-Baire. By Theorem 5 (ii), $F \circ h_\alpha$ is lower semicontinuous. Since α was arbitrary, the assertion follows.

Clearly, Theorem 7 yields the statements mentioned in the passage before Lemma 4.

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(Oblatum 14.8. 1981)