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ON RECURSIVE MEASURE OF CLASSES OF RECURSIVE SETS

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Abstract: It is shown that any class of recursive sets $\{\varphi_{h(n)} : n \in N\}$ where h is a function of degree $\underset{\sim}{a}$ such that $\underset{\sim}{a} \cup \underset{\sim}{0}' \not\geq \underset{\sim}{0}''$ has $\underset{\sim}{0}$-measure zero ($\underset{\sim}{0}$-measure is a recursive analogue of the product measure on 2^N).

Key words: Recursive set, recursively enumerable set, degree.

Classification: 03D30, 03F60

It is known that the recursive sets are not uniformly recursive. C. Jockusch [4, Theorem 9] observed that there is a function h of degree $\leq \underset{\sim}{a}$ such that $\varphi_{h(0)}, \varphi_{h(1)}, \dots$ are precisely the recursive sets iff $\underset{\sim}{a} \cup \underset{\sim}{0}' \geq \underset{\sim}{0}''$. In this paper we prove that any class of recursive sets $\{\varphi_{h(n)} : n \in N\}$ where h is a function of degree $\leq \underset{\sim}{a}$ such that $\underset{\sim}{a} \cup \underset{\sim}{0}' \not\geq \underset{\sim}{0}''$ even has $\underset{\sim}{0}$-measure zero. The concept of $\underset{\sim}{0}$-measure is an effective analogue of the product measure on 2^N. It was introduced by O. Demuth [1] for constructive real numbers and plays the important role in constructive mathematical analysis (see, e.g., [2]).

Our notation and terminology are standard. In particular we use the letters i,j,k,n for elements of $N = \{0,1,\dots\}$. We identify subsets of N with their characteristic functions.

A string is a finite sequence of 0 's and 1 's. Strings may also be viewed as functions from finite initial segments of N into $\{0,1\}$. We use the letters σ, τ for strings, $\ell h(\sigma)$ is the length of σ and $\sigma * \tau$ is the string which results from concatenating σ and τ . A subset A of N extends σ ($A \supseteq \sigma$) if the characteristic function of A extends σ . We assume that the set of all strings is effectively Gödel-numbered so that we can apply notions of recursion theory to strings. For functions f, g we say that f dominates g if $f(n) \geq g(n)$ for all but finitely many n. Let φ_n be the n-th partial recursive function in some standard enumeration of all partial recursive functions.

We shall use the Martin's result [6] that there is a function f of degree $\underset{\sim}{a}$ which dominates every recursive function iff $\underset{\sim}{a}' \geq \underset{\sim}{0}''$. We shall also use the following straightforward modification of the result.

<u>Lemma</u>: For any degree $\underset{\sim}{b}$ and for any class $\mathcal{A} = \{\varphi_{h(n)} : n \in N\}$ of recursive functions where h is a function of degree $\leq \underset{\sim}{b}'$ there is a function f of degree $\leq \underset{\sim}{b}$ which dominates all functions of $\mathcal{A}$.

We shall use a special case of the concept of $\underset{\sim}{0}$-measure (see [1]).

<u>Definition</u>: A class $\mathcal{A}$ of subsets of N has $\underset{\sim}{0}$-measure zero if there exist a recursive sequence $R_0, R_1, \dots$ of r.e. sets of strings and a recursive sequence $y_0, y_1, \dots$ of constructive real numbers (i.e. recursive reals) such that for every n

1) the real number $\sum_{\sigma \in R_n} 2^{-\ell h(\sigma)}$ is equal to y_n and $y_n \leq 2^{-n}$,

2) for any set A, $A \in \mathcal{A}$, there is a string σ, $\sigma \in R_n$, such that $\sigma \subseteq A$.

It should be noted two important facts in the definition:

i) $\sum_{\sigma \in R_n} 2^{-\ell h(\sigma)}$ is required to be equal to a constructive real number for every n,

ii) $y_0, y_1, \ldots$ is required to form a recursive sequence.

Zaslavskij and Cejtin [8] proved that the class of all recursive sets has $\underset{\sim}{O}$-measure equal to 1. More information on the role of $\underset{\sim}{O}$-measure and some survey of constructive mathematical analysis can be found in [2].

<u>Theorem</u>: If $\underset{\sim}{a}$ is a degree such that $\underset{\sim}{a} \cup \underset{\sim}{O}' \neq \underset{\sim}{O}''$ then any class of recursive sets $\{\varphi_{h(n)} : n \in N\}$ where h is a function of degree $\leq \underset{\sim}{a}$ has $\underset{\sim}{O}$-measure zero.

<u>Proof</u>. It follows from [8] or from [5] that there is a r.e. set S_0 of strings such that

1) $\sum_{\sigma \in S_0} 2^{-\ell h(\sigma)}$ is less than $\frac{1}{2}$,

2) for every recursive set A there exists a string σ, $\sigma \in S_0$, such that $\sigma \subseteq A$,

(i.e. there is a recursive binary tree T without infinite recursive branches such that the usual product measure on 2^N of the class of all infinite branches of T is greater than $\frac{1}{2}$). It should be noted that the real number $\sum_{\sigma \in S_0} 2^{-\ell h(\sigma)}$ is recursive in $\emptyset'$ but it cannot be equal to any constructive real number (see [8]).

Let $S_0, S_1, \ldots$ be a recursive sequence of r.e. sets of strings such that for every n $S_{n+1} = \{\sigma * \tau : \sigma \in S_n \ \& \ \tau \in S_0\}$.

Let $\{\sigma_{n,k} : k \in N\}$ be a recursive enumeration of S_n for every n (all S_n are, of course, infinite). It is easy to verify that $\sum_{\sigma \in S_n} 2^{-\ell h(\sigma)} < 2^{-(n+1)}$ for all n.

Further, for any recursive set A we can effectively find a recursive function α such that for all n $A \supseteq \sigma_{n,\alpha(n)}$. So, let g be a recursive function such that if φ_n is a recursive set then $\varphi_n \supseteq \sigma_{k,\varphi_{g(n)}(k)}$ for all k, n. Now let $\underset{\sim}{a}$ be a degree such that $\underset{\sim}{a} \cup \underset{\sim}{O}' \neq \underset{\sim}{O}''$ and h be a function of degree $\leq \underset{\sim}{a}$ such that $\{\varphi_{h(n)} : n \in N\}$ is a class of recursive sets. We use the function g described above to form the class of recursive functions $\mathcal{B} = \{\varphi_{gh(n)} : n \in N\}$. The function gh is obviously of degree $\leq \underset{\sim}{a}$. By the theorem of Friedberg [3] (or [7] § 13.3) there is a degree $\underset{\sim}{b}$ such that $\underset{\sim}{b}' = \underset{\sim}{a} \cup \underset{\sim}{O}'$. By the lemma there is a function f of degree $\leq \underset{\sim}{b}$ which dominates all functions of the class $\mathcal{B}$. Since $\underset{\sim}{b}' \neq \underset{\sim}{O}''$, there is a recursive function δ which f fails to dominate. Thus, for all n, $\varphi_{gh(n)}(k) \leq \delta(k)$ for infinitely many k. By the properties of g we have $\varphi_{h(n)} \supseteq \sigma_{k,\varphi_{gh(n)}(k)}$ for all k, n. Let $R_0, R_1, \ldots$ be a recursive sequence of r.e. sets of strings such that for every n $R_n = \{\sigma_{k,j} : k \geq n \ \& \ j \leq \delta(k)\}$. It follows that for all i, n there is a string $\sigma \in R_n$ such that $\varphi_{h(i)} \supseteq \sigma$. Further, it is easy to construct a recursive sequence of constructive real numbers $y_0, y_1, \ldots$ such that for all n $\sum_{\sigma \in R_n} 2^{-\ell h(\sigma)}$ is equal to y_n and $y_n \leq 2^{-n}$.

Thus, the class $\{\varphi_{h(n)} : n \in N\}$ has $\underset{\sim}{O}$-measure zero.

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