COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
20, 1 (1979)

ON ORDER TOPOLOGY OF SPACES HAVING UNIFORM LINEARLY ORDERED BASES

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Abstract: It is shown that a dense in itself topological space X which has a uniformity with a linearly ordered (with respect to star-refinements) base of uncountable cofinality is an ordered topological space.

Key words: Order topology, linearly ordered base of uniformity.

AMS: 54F05, 54E15

A class of topological spaces which have uniformities with linearly ordered bases (shortly, with uniform l.o. bases) contains all metrizable spaces. The topology of a metrizable space is induced by a uniformity with a countable base linearly ordered (with respect to the star-refinements of coverings). Herrlich [1] has proved (and Lynn [3] for separable metric spaces) that for each metric space X with dim X = = 0, the topology of X is induced by a linear order. Our result can be treated as an extension of the results of Herrlich and Lynn. If a space X has a uniformity with l.o. bases then X is metrizable or X is paracompact, dim X = 0, and X is a dense subspace of the limit of an inverse system over well-ordered set of discrete spaces [2]. Consequently, if X is dense in itself, then the topology of X is an order topology.

If a space X with a uniform l.o. base of uncountable cofinality has "many" isolated points then we do not know if it is true that the topology of X is an order topology. We can apply proof that such a space is a GO-space, i.e. a subspace of an order space. The special case, every topological group with linearly ordered base of neighborhoods of the neutral element is orderable, was proved in [4].

Lemma 1 [2]. If a space X has a uniform l.o. base B of an uncountable cofinality, of $B > \aleph_0$, then for each family $\mathcal{R}$ of open sets with card $\mathcal{R} <$ of B , the intersection $\cap \mathcal{R}$ is an open set.

Proof. Let $x \in \cap \mathcal{R}$. For each $G \in \mathcal{R}$ let us choose a $P_G \in B$ such that $st(x,P_G) \subset G$. Since card $\{P_G : G \in \mathcal{R}\} <$ of B, there exists $P \in B$ such that $P \succ P_G$ ($P \succ Q$ means that P is a refinement of Q) for each $G \in \mathcal{R}$. Hence $st(x,P) \subset \cap \mathcal{R}$. Thus $\cap \mathcal{R}$ is an open set.

From Lemma 1 it follows that if a space X has a uniform l.o. base B with cf $B > \aleph_0$, then each $\mathcal{G}_\delta$ subset is open in X, consequently, dim X = 0 [2]. Indeed, let $\{V_i : i = 1,\dots,k\}$ be a finite functionally open covering of the space X. There exists a functionally closed covering $\{F_i : i = 1,\dots,k\}$ such that $F_i \subset V_i$, $i = 1,\dots,k$. Each F_i is a $\mathcal{G}_\delta$ set, so it is clopen set. Put $U_1 = F_1$ and $U_j = F_j - \bigcup\{U_i : i < j\}$. The family $\{U_i : i = 1,\dots,k\}$ is an open covering of X, $U_i \subset V_i$, $U_i \cap U_j = \emptyset$ for i j, $i,j = 1,\dots,k$. Thus dim X = 0.

Lemma 2 [2]. Each topological space which has a uniform l.o. base is paracompact.

Proof. Since each linearly ordered set contains a cofinal and well-ordered subset we may assume that $B = \{P_\alpha : \alpha < \gamma\}$, $P_\alpha \succ_* P_\beta$ iff $\alpha > \beta$, is a well-ordered with respect to the star-refinements uniform base for X. Let P be an open covering of X. Define $Q = \{ st(x,P_{\alpha+2}) : st(x,P_\alpha) \subset u,\ u \in P,\ x \in X\}$. The covering Q is a star-refinement of P. This implies that X is a paracompact space.

Lemma 3 [2]. If a space X has a uniform l.o. base with $cf\ B > \aleph_0$, then it has a uniform l.o. base B' consisting of open coverings of order 1.

Proof. Let $B = \{P_\alpha : \alpha \prec \gamma\}$, $\gamma = cf\ B$, be a well-ordered uniform base on X. Define zero-dimensional base $B' = \{Q_\alpha : \alpha < \gamma\}$. Since dim X = 0 and X is paracompact, there exists an open covering $Q_1 \succ_* P_1$ and Q_1 is of order 1. Let us assume that Q_α, $\alpha < \beta < \gamma$, are defined. By Lemma 1, there exists an open covering P such that $P \succ_* P_\beta$ and $P \succ_* Q_\alpha$, $\alpha < \beta$. Let $Q_\beta \succ P$ be an open covering of order 1.

Theorem. If a dense in itself space X has a uniform l.o. base of uncountable cofinality, then there exists a linear order on X inducing the topology of the space X.

Proof. Notice that T is an infinite set, then there is a linear order $\prec$ on T such that each $x \in T$ has elements x - 1 and m + 1 in a sense of the discrete order $\prec$. Indeed, let $\dashv$ be an arbitrary linear order on T, then the lexicographic order on $T \times Z$, where Z is the set of integers is a discrete order. Since card T = card $(T \times Z)$, hence T has a discrete order without the first and the last element.

Let $B = \{P_\alpha : \alpha < \gamma\}$, $\gamma = cf\ B$, $P_\alpha \succ_* P_\beta$ iff $\alpha \geqslant \beta$, be a uniform well-ordered base consisting of open coverings of order 1. For each $x \in X$ put $x(\alpha) = u \in P_\alpha$, such that $x \in u$, and for each $u \in P_\alpha$ let $\pi(u) = \{v \in P_{\alpha+1} : v \in u\}$. Since X has bo isolated point, without loss of generality we may assume that for each $u \in P_\alpha$, card $\pi(u) \geq \aleph_0$.

Now, assume that for each $u \in P_\alpha$, $\alpha < \gamma$, it is chosen a discrete order $<$ (without the first and the last element) on $\pi(u)$ and let us assume that it is given a discrete order $<$ on each P_β, where $\beta < \gamma$ is a limit ordinal.

Define a linear order on X. For each $x, y \in X$ let us put $x < y$ iff $x(\alpha) < y(\alpha)$, where $\alpha = \min\{\beta < \gamma : x(\beta) \neq y(\beta)\}$.

Now, we shall show that the topology induced by the order $<$ is equal to the topology of the space X. Notice that $B^* = \bigcup B$ is a base for the topology of X. Let $z \in u \in P_\alpha$, $\alpha < \gamma$. There exist $z(\alpha + 1) - 1$, $z(\alpha + 1) + 1 \in \pi(u)$. Choose $x, y \in X$ such that $x(\alpha + 1) = z(\alpha + 1) - 1$, $y(\alpha + 1) = z(\alpha + 1) + 1$. Notice that $\langle x,y \rangle \subset u$. Now, consider an interval $\langle x,y \rangle$ and $z \in \langle x,y \rangle$. There is the least α, $\beta < \gamma$ such that $x(\alpha) \in z(\alpha)$ and $z(\beta) < y(\beta)$. If $\alpha \leq \beta$, then $z(\beta) \subset \langle x,y \rangle$. If $\beta \leq \alpha$, then $z(\alpha) \subset \langle x,y \rangle$. But $z(\alpha)$, $z(\beta)$ are open neighbourhoods of the point z. Thus the topologies are equal.

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(Oblatum 14.3.1978)