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A NOTE ON CLOSE-TO-NORMAL STRUCTURE

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__Abstract__: Necessary and sufficient conditions under which a convex subset of a Banach space possesses a close-to-normal structure are established.

__Key words__: Close-to-normal structure, convex sets, Banach spaces, fixed point.

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Let X be a real Banach space. A convex subset K of X is said to have a close-to- normal structure if for any bounded closed convex subset H of K with the diameter $\delta(H) > 0$, there exists x in H such that $\|x - y\| < \delta(H)$ for all y in H. It is well-known that the notion of close-to-normal structure is useful in the fixed point theory. For instance, C.S. Wong [1] has proved that every Kannan map on a weakly compact convex subset K of X has a unique fixed point if K has a close-to-normal structure. (A self map T on K is a Kannan map if, for all x, y in K,

$\|Tx - Ty\| \le \frac{1}{2} (\|x - Tx\| + \|y - Ty\|)$.)

The purpose of this note is to establish some results concerning the close-to-normal structure . Section 1 deals with necessary and sufficient conditions under which a convex subset of a Banach space possesses the close-to-normal

The methods of the proofs of our results are similar to those of M.S. Brodskii and D.P. Milman [2] and of T.C. Lim [3]. Section 2 solves the following problem which naturally arises with respect to the result of C.S. Wong mentioned above: Every weakly compact convex subset of a Banach space has a close-to-normal structure. Simple examples are given to show the independence of these qualities.

1. Some positive results. We shall say that a nonconstant bounded sequence $\{x_n\}_{n=1}^{\infty}$ is a strictly diametral sequence if there is an integer N such that

$$d(x_{n+1}, co(x_1, \ldots, x_n)) = \delta(\{x_n\}_{n=1}^{\infty})$$

for every $n > N$.

Proposition 1. A convex subset of a Banach space has a close-to-normal structure if and only if it contains no strictly diametral sequence.

Proof. Suppose that a convex subset K of a Banach space X contains a strictly diametral sequence $\{x_n\}_{n=1}^{\infty}$. Let $K_o = co(\{x_n\}_{n=1}^{\infty}) \subset K$. If $x_o \in K_o$, then $x_o = \sum_{i=1}^{p} \alpha_i x_i$, $\alpha_i \geq 0$ $\forall$ $i = 1, \ldots, p$; $\sum_{i=1}^{p} \alpha_i = 1$ and $x_o \in co(x_1, \ldots, x_{m-1})$ $\forall$ $m > p$. Since $\{x_n\}_{n=1}^{\infty}$ is a strictly diametral sequence, there is an integer N such that

$$d(x_{n+1}, co(x_1, \ldots, x_n)) = \delta(\{x_n\}_{n=1}^{\infty}), \quad \forall\ n > N.$$

Then

$$\delta(\{x_n\}_{n=1}^{\infty}) \geq \| x_o - x_m \| \geq \delta(\{x_n\}_{n=1}^{\infty}) \quad \forall\ m > p,\ m > N.$$

Hence, with $y_o = x_{p+N} \in K_o$ we have

$$\| x_o - y_o \| = \delta(K_o) = \delta(\{x_n\}_{n=1}^{\infty}).$$

This shows that K does not have a close-to-normal structure.

Suppose now that K does not have a close-to-normal structure. Then K contains a bounded convex subset H such that $d = \delta(H) > 0$ and for each x in H there is an other element y in H such that $\|x - y\| = d$. Choose x_1, x_2 in H such that $\|x_1 - x_2\| = d$. When $\{x_1,\dots,x_n\} \subset H$ have been chosen, we take x_{n+1} in H such that $\|y_n - x_{n+1}\| = d$, where $y_n = \frac{1}{n}\sum_{i=1}^{n} x_i \in H$. Proceeding in this way we get a sequence $\{x_n\}_{n=1}^{\infty} \subset K$. We show that $\{x_n\}_{n=1}^{\infty}$ is a strictly diametral sequence.

Let $x \in co(x_1,\dots,x_n)$ be arbitrary, $x = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \geq 0$ $\forall$ $i = 1,\dots,n$; $\sum_{i=1}^{n} \alpha_i = 1$. Let $\alpha = \max(\alpha_1,\dots,\alpha_n)$. We have:

$$y_n = \sum_{i=1}^{n} \frac{\alpha_i x_i}{n\alpha} - \sum_{i=1}^{n} \frac{\alpha_i x_i}{n\alpha} + \sum_{i=1}^{n} \frac{x_i}{n} = \frac{x}{n\alpha} + \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{\alpha_i}{n\alpha}\right) x_i;$$

$$\frac{1}{n\alpha} + \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{\alpha_i}{n\alpha}\right) = 1 \text{ and } \frac{1}{n} - \frac{\alpha_i}{n\alpha} \geq 0 \quad \forall\, i = 1,\dots,n.$$

Then

$$d = \|y_n - x_{n+1}\| \leq \frac{1}{n\alpha}\|x - x_{n+1}\| + \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{\alpha_i}{n\alpha}\right) \|x_i - x_{n+1}\|$$

$$\leq \frac{1}{n\alpha}\|x - x_{n+1}\| + d\left(1 - \frac{1}{n\alpha}\right).$$

Hence

$$\frac{d}{n\alpha} \leq \frac{1}{n\alpha}\|x - x_{n+1}\|$$

implies that

$$\|x - x_{n+1}\| = d.$$

Since $x \in co(x_1,\dots,x_n)$ is arbitrary it follows that

$$d(x_{n+1}, co(x_1,\dots,x_n)) = \inf_{x \in co(x_1,\dots,x_n)} \|x - x_{n+1}\| = d, \quad \forall\, n.$$

Thus $\{x_n\}_{n=1}^{\infty}$ is a strictly diametral sequence in K. This completes the proof.

Proposition 2. A convex subset K of a Banach space has a close-to-normal structure if and only if it does not contain a sequence $\{x_n\}_{n=1}^{\infty}$ such that for some $c > 0$, $\| x_n - x_m \| = c$, $\| x_{n+1} - \bar{x}_n \| = c$, for all $n \geq 1$, $m \geq 1$, where $\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$.

Proof. Suppose that K does not have a close-to-normal structure. Then there is a bounded convex subset H of K such that $\delta(H) > 0$ and for every $x \in H$ there is a $y \in H$ such that $\| x - y \| = \delta(H)$. By induction we construct a nonconstant sequence $\{x_n\}_{n=1}^{\infty} \subset H$ as follows: Take $x_1, x_2 \in H$ such that $\| x_1 - x_2 \| = \delta(H)$. Let $x_1, \dots, x_n \in H$ be constructed with the properties that

$$\| x_i - x_k \| = \delta(H), \quad \forall i,k = 1,2,\dots,n \text{ and}$$

$$\| x_{k+1} - \bar{x}_k \| = \delta(H), \quad \forall k = 1,2,\dots,n-1.$$

We choose $x_{n+1} \in H$ such that $\| x_{n+1} - \bar{x}_n \| = \delta(H)$. Now we show that with this x_{n+1} we have $\| x_{n+1} - x_i \| = \delta(H)$ $\forall i = 1,\dots,n$. Indeed, since $\| x_{n+1} - \bar{x}_n \| = \delta(H)$,

$$\delta(H) \equiv n \cdot \frac{\delta(H)}{n} \geq \sum_{i=1}^{n} \frac{\| x_{n+1} - x_i \|}{n} \geq \| x_{n+1} - \bar{x}_n \| = \delta(H).$$

From this it follows that

$$\frac{1}{n} \sum_{i=1}^{n} \| x_{n+1} - x_i \| = \delta(H).$$

Hence

$$\| x_{n+1} - x_i \| = \delta(H), \quad \forall i = 1,\dots,n.$$

So the sequence $\{x_n\}_{n=1}^{\infty} \subset H$ satisfies the condition of the Proposition 2 with $c = \delta(H)$.

On the contrary, assume that K contains a sequence $\{x_n\}_{n=1}^{\infty}$ satisfying the condition of the Proposition 2. Let $x \in co(x_1,\dots,x_n)$. Then

$$x = \sum_{i=1}^{n} \lambda_i x_i ; \quad \lambda_i \geq 0 \quad \forall\, i = 1,\dots,n; \quad \sum_{i=1}^{n} \lambda_i = 1.$$

Let

$$\lambda = \max(\lambda_1,\dots,\lambda_n),$$

$$\gamma_0 = n\lambda ,$$

$$\gamma_i = \lambda_i - \lambda , \quad \forall\, i = 1,\dots,n.$$

We have that

$$0 < \gamma_0 \leq n;$$

$$\gamma_i \leq 0 \quad \forall\, i = 1,\dots,n; \text{ and}$$

$$\sum_{i=1}^{n} \gamma_i = 1.$$

One can write

$$x = \sum_{i=1}^{n} (\lambda_i - \lambda + \lambda)x_i = n\lambda . \sum_{i=1}^{n} \frac{x_i}{n} + \sum_{i=1}^{n} (\lambda_i - \lambda)x_i =$$

$$= \gamma_0 \bar{x}_n + \sum_{i=1}^{n} \gamma_i x_i .$$

Hence,

$$\| x_{n+1} - x \| \leq \sum_{i=1}^{n} \lambda_i \| x_{n+1} - x_i \| = c \text{ and}$$

$$\| x_{n+1} - x \| \geq \| \gamma_0 (x_{n+1} - \bar{x}_n) \| - \sum_{i=1}^{n} \| \gamma_i (x_{n+1} - x_i) \| =$$

$$= \gamma_0 \| x_{n+1} - \bar{x}_n \| + \sum_{i=1}^{n} \gamma_i \| x_{n+1} - x_i \| = c .$$

It follows that $\| x_{n+1} - x \| = c \quad \forall\, n, \ \forall\, x \in co(x_1,\dots,x_n)$. Hence

$$d(x_{n+1}, co(x_1,\dots,x_n)) = c = \delta(\{x_n\}_{n=1}^{\infty}).$$

Thus $\{x_n\}_{n=1}^{\infty}$ is a strictly diametral sequence in K and hence K does not have a close-to-normal structure by Proposition 1. The proposition is proved.

2. Examples. In the sequel we shall always denote by Γ some uncountable set of indices. If X is a space of real-valued functions on Γ which is defined in terms of unconditional convergence, then we denote by K [X] the bounded, convex and closed set

$$\{\{x_\alpha\}_{\alpha\in\Gamma} \in X: x_\alpha \geq 0 \ \forall\alpha \in \Gamma \ , \sum_{\alpha\in\Gamma} x_\alpha \leq 1\} .$$

For the definitions of well-known spaces $\ell^p(\Gamma)$, $c_0(\Gamma)$ with their customary norms see [4].

Example 1. (1.1) The set $K[\ell^2(\Gamma)] \subset \ell^2(\Gamma)$ is weakly compact and possesses a close-to-normal structure.

Since $\ell^2(\Gamma)$ is uniformly convex, $K[\ell^2(\Gamma)]$ is weakly compact and has normal structure. It is obvious that a convex set K has a close-to-normal structure if it has normal structure.

(1.2) The set $K[\ell^1(\Gamma)] \subset \ell^1(\Gamma)$ is not weakly compact and it has no close-to-normal structure.

$K[\ell^1(\Gamma)]$ is not weakly compact since the sequence $\{e_n\}_{n=1}^{\infty} \subset K[\ell^1(\Gamma)]$, $e_n = (0,\dots,1 , 0,\dots)$ contains no convergent subsequence. On the other hand, let

$$H = \{x = \{x_\alpha\}_{\alpha\in\Gamma} \in K[\ell^1(\Gamma)] : \sum_{\alpha\in\Gamma} x_\alpha = 1\} .$$

Then H is a bounded, convex and closed subset of $K[\ell^1(\Gamma)]$ with $\delta(H) = 2$. If $x = \{x_\alpha\}_{\alpha\in\Gamma} \in H$, there is at least one $\alpha_0 \in \Gamma$ such that $x_{\alpha_0} = 0$. Let $y = \{y_\alpha\}_{\alpha\in\Gamma} \in H$ such that

$$y_\alpha = \begin{cases} 0 \text{ if } \alpha \in \Gamma, \alpha \neq \alpha_0 \\ 1 \text{ if } \alpha = \alpha_0 \end{cases}$$

Then $y \in H$ and $\|x - y\| = 2 = \delta(H)$. This shows that K has no close-to-normal structure.

(1.3) The set $K[c_0(\Gamma)] \subset c_0(\Gamma)$ is weakly compact which has no close-to-normal structure.

If $\{y^{(n)}\}_{n=1}^{\infty} \subset K[c_0(\Gamma)] = \{x = \{x_\alpha\}_{\alpha \in \Gamma} \in c_0(\Gamma):$

$$: x_\alpha \geq 0 \ \forall \alpha \ , \sum_{\alpha \in \Gamma} x_\alpha \leq 1\},$$

it is not difficult to see that there is a $y \in K[c_0(\Gamma)]$ and a subsequence $\{y^{(n_k)}\}_{k=1}^{\infty}$ of $\{y^{(n)}\}_{n=1}^{\infty}$ such that $\{y^{(n_k)}\}_{k=1}^{\infty}$ converges to y along co-ordinates (by application of the diagonal method). Since $c_0^*(\Gamma) \cong \ell^1(\Gamma)$, it follows that $y^{(n_k)} \xrightarrow{w} y$ as $k \longrightarrow \infty$. Thus $K[c_0(\Gamma)]$ is weakly compact.

On the other hand, for each $x \in K[c_0(\Gamma)]$ let $y = \{y_\alpha\}_{\alpha \in \Gamma}$ be defined as in (1.2). Then $\|x - y\| = 1 = \sigma(K[c_0(\Gamma)])$. Thus $K[c_0(\Gamma)]$ has no close-to-normal structure.

Example 2. M.M. Say [5] has proved that there exists an equivalent norm $|||\cdot|||$ of $c_0(\Gamma)$ which is strictly convex. Let K be the closed unit ball in $\langle c_0(\Gamma), |||\cdot||| \rangle$. Then K has a close-to-normal structure. (It is easy to prove that every bounded closed convex subset of a strictly convex Banach space has a close-to-normal structure.) But K is not weakly compact because $c_0(\Gamma)$ is not reflexive.

R e f e r e n c e s

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