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(P)-SETS, QUASIPOLYHEDRA AND STABILITY

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Abstract: In this paper the property (P) of convex subsets of normed linear spaces defined in [7] is characterized in terms of the relative openness of affine maps. As an immediate consequence we obtain that any finite dimensional compact convex (P)-set K is stable, that is (see e.g. [4]) the midpoint mapping $(x,y) \longrightarrow \frac{1}{2}(x+y)$ is relatively open on $K \times K$. Also, we characterize in the class of normed linear spaces l_1-products which are (P)-spaces.

Key words: Normed linear space, (P)-set, stable set, quasipolyhedral set.

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If it is not stated otherwise, our notation and terminology is that of [5].

Let X, Y be topological spaces, $f: X \longrightarrow Y$ a mapping, $A \subset X$ a subset and $x \in A$. The mapping f is said to be relatively open on A in x if f maps each neighbourhood of x in A onto a neighbourhood of f(x) in f(A). The mapping f is relatively open on A [relatively open respectively] if f is relatively open on A in each $x \in A$ [f is relatively open on X].

Brown [3] characterized normed linear spaces for which the metric projections onto all finite dimensional subspaces are lower semicontinuous and called them (P)-spaces.

For a list of (P)-spaces we refer the reader to [2].

According to Wegmann [7] a normed linear space X is a (P)-space if and only if the closed unit ball K of X has the property (P), i.e.: for any $x \in K$ and $z \in K$ such that $x + z \in K$ there exists a neighbourhood U of x in K and $c > 0$ such that $y + cz \in K$ for any $y \in U$.

We present here

(1) **Theorem.** Let K be a closed bounded convex subset of a normed linear space X. Then K has the property (P) if and only if for any normed linear space Y and any relatively open linear mapping $T: X \longrightarrow Y$ such that $\dim T_{-1}(0) < +\infty$, T is relatively open on K.

Before proving we formulate

(2) Lemma. Let K be a closed convex subset of a normed linear space X. Then K has the property (P) if and only if K has the following property (we denote if (P_1)): for any $x \in K$ and $z \in K$ such that $x + z \in K$ and any $\varepsilon > 0$ there exists a neighbourhood U of x in K such that $y + (1-\varepsilon)z \in K$ for any $y \in U$.

Proof. Suppose that K satisfies the condition (P) but not the condition (P_1). Thus there exists some $x_0 \in K$ and $z \in X$ such that

(i) $$x_0 + z \in K$$

and a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of K such that x_n tends to x_0 but for $s_n = \sup\{t \geq 0; x_n + tz \in K\}$ there is

(ii) $$\limsup_{n} s_n = s < 1.$$

By choosing a subsequence we can suppose that s_n converges to s. Then for $u_n = x_n + (1-n^{-1})s_n z$ we have $u_n \in K$ by definition of s_n and u_n converges to $x_0 + sz$. By virtue of (i),(ii) and the property (P) of K (applied to $x = x_0 + sz$) there is $c > 0$ such that $u_n + c(1-s)z \in K$ for large n which is the same as $x_n + [(1-n^{-1})s_n + c(1-s)]\, z \in K$. However (ii) implies $(1-n^{-1})s_n + c(1-s) > s$ for large n which contradicts the definition of s_n.

<u>The proof of Theorem (1)</u>. Let K have the property (P), T be as in (1) and $x_0 \in K$ be arbitrary. Suppose T is not relatively open on K in x_0 so that there exists a neighbourhood U of x_0 in K and a sequence $x_n \in K$ such that $T(x_n)$ tends to $T(x_0)$ but $T(x_n)$ has no inverse image in U for any $n \geq 1$.

Since T is relatively open on X there exist $\hat{x}_n \in X$ such that $T(\hat{x}_n) = T(x_n)$ and $\hat{x}_n$ converges to x_0. As $T_{-1}(0)$ is finite dimensional we can suppose $\hat{x}_n - x_n$ to be converging to some $z \in T_{-1}(0)$, hence x_n converges to $x_0 - z \in K$ (K is closed). By virtue of Lemma (2) we can apply the property (P_1) to $x = x_0 - z$ so that $x_n + c_n z \in K$ for some sequence c_n converging to one. The sequence $x_n + c_n z$ converges to x_0 but $x_n + c_n z$ is an inverse image of $T(x_n)$ in K, a contradiction.

For proving the other implication suppose $x \in K$ and $z \in X$ be such that $x + z \in K$. Of course we can suppose $z \neq 0$. Denote $\varepsilon = \frac{1}{3}\|z\|$, N the linear span of z and $T: X \longrightarrow X/N$ the factorization mapping. Since T is relatively open on K by our assumptions the image of the ε-neighbourhood of $x + z$ in K contains a δ-neighbourhood of $T(x+z) = T(x)$ in T(K) for some $0 < \delta < \varepsilon$.

Let U be δ-neighbourhood of x in K. Then for any $y \in U$ we have $\|T(y) - T(x)\| < \delta$ since $\|T\| = 1$. Hence T(y) has an inverse image u in K such that $\|u - (x+z)\| < \varepsilon$. Of course $u = y + cz$ for some constant c because of the definition of T.

Hence $\| cz - z \| < \varepsilon + \| x-y \|$ so that $3\varepsilon |1-c| < \varepsilon + \delta < 2\varepsilon$ which implies $c > \frac{1}{3}$. Thus $y + \frac{1}{3}z \in K$ and the proof is finished.

(3) Corollary. Let K be a closed bounded convex subset of a finite dimensional space such that K has the property (P). Then K is stable (see the introduction).

Proof. The subset $K \times K$ of $X \times X$ is easily seen to have the property (P).

For example any finite dimensional polyhedron of any convex body the boundary of which contains no non-trivial segment has the property (P) (cf. [3] and [7]). Also any (QP)-space in the sense of [1] is a (P)-space ([7]).

We present here a definition of a (QP)-space which is equivalent to that of [1], however more convenient for our aims.

(4) Definition. Let X be a normed linear space, $K \subset X$ a convex subset and $x \in K$. We shall say that K is (qp) in x (quasipolyhedral) if there exists $\delta > 0$ such that if $x + h \in K$ for some $h \in X \setminus \{0\}$, then $x + \delta \frac{h}{\|h\|} \in K$. We shall say that K is (qp) if it is (qp) in any $x \in K$. A normed linear space X is said to be a (QP)-space if the closed unit ball of X is (qp).

It can be seen easily that a convex set K is (qp) if and only if it is locally conic in the sense of [6].

Clearly (closed) halfspace is (qp) and the intersection of a finite number of (qp)-sets is again (qp). Compact (qp)-sets are exactly finite dimensional polyhedrons since the extreme points of a (qp)-set K have clearly no cluster point in K.

For any set I the space $c_0(I)$ is a (QP)-space and also the product of (QP)-spaces in the sense of c_0 is again a (QP)-space ([1]).

Now we formulate

(5) Theorem. Let $\{X_i\}_{i\in I}$ be a family of normed linear spaces, card $I>1$, dim $X_i \geq 1$ for any $i\in I$ and let X be the product of $\{X_i\}_{i\in I}$ in the sense of $l_1(I)$. Then X is a (P)-space if and only if the set I is finite and X_i is a (QP)-space for any $i \in I$.

Proof. If the set I is finite and X_i is a (QP)-space for any $i\in I$, then X is a (QP)-space ([1]) and thus X is a (P)-space ([7]).

On the other hand suppose X is a (P)-space. Then the set I is finite ([2]). The rest of the proof is an elementary calculus using the definitions.

Thus Theorem (5) gives examples of normed linear spaces which are not (P)-spaces.

As to the stability of (qp)-sets we have

(6) Proposition. Any bounded (qp)-subset of a normed linear space is stable.

The proof follows immediately from

(7) Lemma. Let X, Y be normed linear spaces, $T:X \longrightarrow Y$ a linear mapping, $K\subset X$ a bounded convex set and $x\in K$. Suppose T(K) is (qp) in T(x). Then T is relatively open on K in x.

Proof. Denote $y = T(x)$. Let $\delta > 0$ be such that $y + \delta \| h \|^{-1} h \in T(K)$ whenever $y + h \in T(K)$ for some $h \neq 0$. We can suppose the diameter of K is positive. Let $\varepsilon > 0$ be arbitrary such that $\varepsilon <$ diam K. We show that T maps ε-neighbourhood of x in K onto at least α-neighbourhood of T(x) in T(K) for $\alpha = \varepsilon \delta$ (diam K)$^{-1}$.

Let $v \in T(K)$ be within α from y, $v \neq y$. Then for $w = y + \delta \| v-y \|^{-1}(v-y)$ we have $w \in T(K)$ by the definition of δ. Let x_w be an inverse image of w in K. Then $x_v = x + \delta^{-1} \| v-y \| (x_w - x)$ is an inverse image of v in K since $\delta^{-1} \| v-y \| < \delta^{-1} \alpha = \varepsilon(\text{diam } K)^{-1} < 1$. However $\| x_v - x \| \leqq \delta^{-1} \alpha$ diam $K < \varepsilon$.

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