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PSEUDO-CONTRACTIVE MAPPINGS AND THE LERAY-SCHAUDER BOUNDARY CONDITION

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Abstract: Let X be a Banach space and D a bounded closed subset of X with $0 \in \text{int}(D)$. A mapping $T:D \to X$ such that $(\lambda-1)\|x-y\| \leq \|(\lambda I-T)(x)-(\lambda I-T)(y)\|$ for all $x,y \in D$, $r>0$, is called pseudo-contractive, while T is said to be nonexpansive if $\|T(x)-T(y)\| \leq \|x-y\|$, $x,y \in D$. It is well-known that if T is nonexpansive, then the Leray-Schauder condition: $T(x) \neq \lambda x$ for $x \in \partial D$, $\lambda > 1$, is sufficient to guarantee that $\inf\{\|x-T(x)\| : x \in D\} = 0$. This result is extended here to the wider class of continuous pseudo-contractive mappings under the weaker Leray-Schauder condition: $T(x) = \lambda x$ for $x \in \partial D$, $\lambda > 1 \Rightarrow T(y) = \mu y$ for some $y \in D$ and $\mu \in [1,\lambda)$. Several related results are also obtained.

Key words and phrases: Banach space, fixed point theorems, pseudo-contractive mappings.

Classification: 47H10, 47H15

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1. Introduction. Let X be a Banach space and D a subset of X. A mapping $T:D \to X$ is said to be k-pseudo-contractive if for each $u,v \in D$ and $\lambda > k$:

(1) $$(\lambda-k)\|u-v\| \leq \|(\lambda I-T)(u)-(\lambda I-T)(v)\| .$$

For $k<1$ ($k=1$) such mappings are said to be strongly pseudo-contractive (resp., pseudo-contractive).

In addition to generalizing the nonexpansive mappings (mappings $T:D \to X$ for which $\|T(x)-T(y)\| \leq \|x-y\|$, $x,y \in D$), the pseudo-contractive mappings are characterized by the important fact that a mapping $T:D \rightarrow X$ is pseudo-contractive if and only if the mapping $f = I-T$ is <u>accretive</u> on D (see Browder [1]; Kato [5]). Because of this and the role played by the accretive mappings in the study of nonlinear evolution equations in Banach spaces, the pseudo-contractive mappings have been widely studied in recent years (e.g., [6],[8],[10],[12],[13]). Our purpose here is to prove new fixed point theorems for continuous pseudo-contractive mappings. For the most part, our results generalize results which are known either for the nonexpansive mappings or for more restricted classes of pseudo-contractive mappings (e.g., the lipschitzian or k-set-contractive pseudo-contractive mappings). We note, however, that our results are obtained under a somewhat weaker boundary condition (condition ($\mathfrak{L}$) below). In this respect our theorems are often new even for the more restricted classes of mappings.

Throughout our discussion, X will denote a Banach space, and for $D \subset X$ we use int(D) to denote the interior of D and $\overline{D}$ to denote the closure of D.

Our original motivation for this paper evolved from the well-known fact if D is bounded and $0 \in \text{int}(D) \subset X$, and if $T:\overline{D} \rightarrow X$ is nonexpansive, then the <u>Leray-Schauder</u> boundary condition

(L-S) $\qquad T(x) \neq \lambda x \quad \text{for} \quad x \in \partial D, \ \lambda > 1$

(∂D denotes the boundary of D) is sufficient to guarantee that $\inf\{\|x-T(x)\| : x \in \overline{D}\} = 0$. This fact is a special case of ideas of Nussbaum [11] (cf. Pétryshyn [12]). After proving some preliminary results in Section 2, we extend the above result in Section 3 to the much wider class of continuous pseudo-contractive mappings under the weaker Leray-Schauder type assumption ($\mathcal{E}$) introduced in Kirk-Morales [7]. Specifically, for $A \subset D$, let $\mathcal{E}_A = \{\lambda > 1 : T(x) = \lambda x$ for some $x \in A\}$. In the results to follow we shall suppose the mapping $T:\bar{D} \to X$ satisfies

$$(\mathcal{E}) \qquad \lambda \in \mathcal{E}_{\partial D} \Longrightarrow \mathcal{E}_{\bar{D}} \cap [1, \lambda) \neq \emptyset.$$

The condition (L-S) is equivalent to $\mathcal{E}_{\partial D} = \emptyset$, the vacuous case of ($\mathcal{E}$).

2. Preliminary results. Our main result in this section is the following.

Theorem 1. Let X be a Banach space, D a closed subset of X with $0 \in \text{int}(D)$, and T a continuous strongly pseudo-contractive mapping of D into X satisfying

$$\text{(i)} \qquad \lambda \in \mathcal{E}_{\partial D} \Longrightarrow \mathcal{E}_{D} \cap [1, \lambda) \neq \emptyset.$$

Then T has a fixed point in D.

We derive Theorem 1 from a sequence of propositions. The first of these collects several important properties of k-pseudo-contractive mappings.

Proposition 1. Let X be a Banach space, D a subset of X and $T:D \to X$ a k-pseudo-contractive mapping, $k > 0$. Set

$$\mathcal{E}_D(\alpha) = \{\lambda > \alpha : T(x) = \lambda x \text{ for some } x \in D\},$$

$$E(\alpha) = \{x \in D : T(x) = \lambda x \text{ for some } \lambda > \alpha\}.$$

Then:

(i) for each $\lambda \in \mathcal{E}_D(k)$ there exists a unique x in D for which $T(x) = \lambda x$;

(ii) if $\lambda \in \mathcal{E}_D(k)$ and $x_\mu \in D$ ($\mu \in \mathbb{R}$) with $T(x_\mu) = \mu x_\mu$,

$$\| x_\lambda - x_\mu \| \leq \| x_\mu \| \, |\lambda - \mu| / (\lambda - k);$$

(iii) $E(\alpha)$ is bounded for each $\alpha > k$;

(iv) the mapping $\psi : \mathcal{E}_D(\alpha) \longrightarrow X$ defined by $\psi(\lambda) = x_\lambda$, where $T(x_\lambda) = \lambda x_\lambda$, is lipschitzian for $\alpha > k$;

(v) if $Tx = kx$ for some $x \in D$, then $E(k)$ is bounded:,

(vi) if $\{x_n\}$ is a bounded sequence in D, $\lambda_n \rightarrow \lambda > k$, and $x_n - \lambda_n^{-1} T(x_n) \rightarrow y$, then $\{x_n\}$ is Cauchy;

(vii) for D closed and T continuous, $\mathcal{E}_D(k)$ is closed in (k, ∞);

(viii) if $0 \in \text{int}(D)$ and T is continuous, there exists $r > 0$ such that $(r, \infty) \subset \mathcal{E}_{\text{int}(D)}(\alpha)$ for all $\alpha \geq k$.

Since most of these properties follow easily from (1) and the others are direct consequences of some of the previous ones, we omit the proofs.

In the next proposition, note that $\mathcal{E}_D = \mathcal{E}_D(1)$ in the notation of Proposition 1.

__Proposition 2.__ Let X be a Banach space, D an open subset of X and $T : D \longrightarrow X$ a continuous pseudo-contractive mapping. Then the set $\mathcal{E}_D$ is open in $(1, \infty)$.

__Proof.__ Let U be an open ball with $\overline{U} \subset D$ and $t \in (0,1)$. Then $(I-tT)(\overline{U})$ is closed (by Proposition 1(vi)). Since, by the above remark, tT is strongly pseudo-contractive it follows, by Theorem 3 of [3], that $(I-tT)(\overline{U})$ is open.

Therefore $\partial[(I-tT)(U)] \subset (I-tT)(\partial U)$.

Suppose now that $\mathcal{E}_D$ is not open. Then there exists $t \in (0,1)$ for which $0 \in (I-tT)(D)$ but for which $0 \notin (I-t_nT)(D)$ for some sequence $\{t_n\} \subset (0,1)$ with $t_n \to t$. Let $0 = x-tTx$ and suppose B is a closed ball centered at x (in D). Then for each n,

$$y_n = x - t_nTx \in (I-t_nT)(B)$$

while $0 \notin (I-t_nT)(B)$. Select $z_n \in \text{seg}[0,y_n] \cap \partial[(I-t_nT)(B)]$. Since t_nT is strongly pseudo-contractive on B,

$$\partial[(I-t_nT)(B)] \subset (I-t_nT)(\partial B).$$

It follows that there exists a point $x_n \in \partial B$ such that $x_n-t_nT(x_n) = z_n$. Since $y_n \to 0$ as $n \to \infty$ and $z_n \in \text{seg}[0,y_n]$, $z_n \to 0$; moreover by assumption, $\{x_n\}$ is bounded, and $\{t_n\}$ is bounded away from zero. Hence $\{T(x_n)\}$ is bounded, proving $x_n-tT(x_n) \to 0$ as $n \to \infty$. Since T is pseudo-contractive, we conclude, by Proposition 1(vi) that $\{x_n\}$ is a Cauchy sequence which must converge, say to $\bar{x} \in \partial B$. Thus $tT(\bar{x}) = \bar{x}$ which, since $\bar{x} \neq x$, contradicts one-to-oneness of I-tT on B.

Finally, we state the last proposition. This is an immediate consequence of Proposition 2 and therefore the proof is omitted.

<u>Proposition 3</u>. Let X be a Banach space, D a closed subset of X and $T:D \to X$ a continuous k-pseudo-contractive mapping. Let $\lambda(\alpha) = \inf \mathcal{E}_D(\alpha)$ for $\alpha \geq k$ and $\mu(\alpha) = \sup(\mathcal{E}_D(k) \cap (k,\alpha))$ for $k < \alpha$ ($\inf \emptyset = \infty$, $\sup \emptyset = -\infty$). Then:

(i) $\lambda(\alpha) \in \mathcal{E}_{\partial D}(k)$ for $\alpha < \lambda(\alpha)$ and

(ii) $\mu(\alpha) \in \mathscr{E}_{\partial D}(k)$ for $-\infty < \mu(\alpha) < \alpha$.

Proof of Theorem 1. Since, by assumption, T is continuous and $0 \in int(D)$, then $\mathscr{E}_D \neq \emptyset$ (by Proposition 1(viii)). Let $\lambda_0 = \inf \mathscr{E}_D$ and suppose $\lambda_0 > 1$. Select a sequence $\{\lambda_n\} \subset \mathscr{E}_D$ with $\lambda_n \to \lambda_0$. By assumption there exists $x_n \in D$ such that $T(x_n) = \lambda_n x_n$ and by the strong pseudo-contractiveness of T, Proposition 1(vi) implies that $x_n \to x \in D$. By continuity of T, $T(x) = \lambda_0 x$. Since $x \in int(D)$ (by (i)), $\lambda_0 \in \mathscr{E}_{int(D)}$. However, by Proposition 3(i) (for $\alpha = 1$), $\lambda_0 \in \mathscr{E}_{\partial D}$, which is a contradiction. Hence $\lambda_0 = 1$, and thus by Proposition 1[(iii), (vi)], we conclude that $1 \in \mathscr{E}_D$, i.e., $T(x) = x$.

3. General results. The results of this section are formulated either in arbitrary Banach spaces or, for stronger conclusions, either in reflexive spaces or spaces in which the domain D of the mapping in question has the fixed-point property relative to nonexpansive self-mappings.

Theorem 2. Let X be a Banach space, D a closed subset of X with $0 \in int(D)$, and $T:D \to X$ a continuous pseudo-contractive mapping of D into X satisfying

(i) $\lambda \in \mathscr{E}_{\partial D} \Longrightarrow \mathscr{E}_D \cap [1, \lambda) \neq \emptyset$, and

(ii) $E = \{x \in D : T(x) = \lambda x$ for some $\lambda > 1\}$ is bounded.

Then $\inf\{\|x - T(x)\| : x \in D\} = 0$.

Proof. Let $\lambda_0 = \inf \mathscr{E}_D$ and assume $\lambda_0 > 1$. Then there exists a sequence $\{\alpha_n\}$ in $[\lambda_0^{-1}, 1)$ with $\alpha_n \to 1$, and by Lemma 2 of [7], $\alpha_n T$ satisfies condition (i) (with the sets $\mathscr{E}_{\partial D}$, $\mathscr{E}_D$ defined relative to $\alpha_n T$). Since $\alpha_n T$ is strongly pseudo-contractive, by Theorem 1 there exists $x_n \in D$

such that $\alpha_n T(x_n) = x_n$. It follows (by (ii)) that $\{x_n\}$ is bounded and thus $\|x_n - T(x_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$, which yields $\inf\{\|x-T(x)\| : x \in D\} = 0$. If $\lambda_0 = 1$, then there exists a sequence $\{\lambda_n\}$ in $\mathcal{E}_D$ such that $\lambda_n \longrightarrow 1$ and $T(x_n) = \lambda_n x_n$ for some $x_n \in D$. Thus $\|x_n - T(x_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$ again, completing the proof.

Theorem 3. Let X be a Banach space, D a closed bounded subset of X with $0 \in \mathrm{int}(D)$, and suppose D has the fixed point property with respect to nonexpansive self-mappings. Suppose $T: D \longrightarrow X$ is a continuous pseudo-contractive mapping which satisfies

(i) $\lambda \in \mathcal{E}_{\partial D} \Longrightarrow \mathcal{E}_D \cap [1, \lambda) \neq \emptyset$;

(ii) $\inf\{\|x-T(x)\| : x \in \partial D\} > 0$.

Then T has a fixed point in D.

Proof. By Theorem 2, $\inf\{\|x-T(x)\| : x \in D\} = 0$. Thus there exists $z \in \mathrm{int}(D)$ such that $\|z-T(z)\| < \|x-T(x)\|$ for all $x \in \partial D$ and it follows from Theorem 1 of [8] that T has a fixed point in D.

Before stating our next theorem, we recall that a mapping $\mu: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is said to be a gauge function if μ is continuous and strictly monotone such that $\mu(0) = 0$ and $\lim_{t\to\infty} \mu(t) = \infty$.

A Banach space X is said to admit a weakly sequentially continuous duality mapping J_μ [2] if, whenever $\{x_n\}$ converges weakly to $x_0 \in X$, then there are functionals $f_n \in J_\mu(x_n)$ which converge weakly in X^* to some $f_0 \in J_\mu(x_0)$, where $J_\mu: X \longrightarrow 2^{X^*}$ is a duality mapping with respect to μ which is defined by

$$J_\mu(x) = \{ j \in X^*: (x,j) = \| x \| \cdot \mu(\| x \|), \| j \| = \mu(\| x \|) \}.$$

Theorem 4. Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J_μ with respect to some gauge function μ. Let D be a closed subset of X with $0 \in \mathrm{int}(D)$, and $T: D \to X$ a continuous pseudo-contractive mapping which satisfies the conditions:

(i) $\lambda \in \mathscr{E}_{\partial D} \Rightarrow \mathscr{E}_D \cap [1, \lambda) \neq \emptyset$;

(ii) $E = \{ x \in D: T(x) = \lambda x$ for some $\lambda > 1 \}$ is bounded.

Then T has a fixed point in D.

Proof. Assume $\lambda_0 > 1$ with $\lambda_0 = \inf \mathscr{E}_D$. Then there exists a sequence $\{\alpha_n\}$ in $[\lambda_0^{-1}, 1)$ with $\alpha_n \to 1$, and as before, by Lemma 2 of [7], $\alpha_n T$ satisfies condition (i) (with the sets $\mathscr{E}_{\partial D}$, $\mathscr{E}_D$ defined relative to $\alpha_n T$). Since $\alpha_n T$ is strongly pseudo-contractive, we conclude by Lemma 2.7 of [10] that T has a fixed point in D. If $\lambda_0 = 1$, the conclusion follows from Lemma 2.7 again.

4. Uniformly convex spaces. In this section, we first prove a theorem concerning existence of fixed points for continuous pseudo-contractive mappings in uniformly convex spaces. Secondly (Theorem 6), we note that a theorem of Martin [9] implies that a result of Petryshyn ([13, Theorem 4.4]) for pseudo-contractive mappings which are also k-set-contractions is actually true for continuous pseudo-contractive mappings and, moreover, does not require convexity of the domain relative to which the Leray-Schauder assumption is imposed.

The next proposition is an immediate consequence of Theorem 5 of [8] and Theorem 2 of section 3.

Proposition 4. Let X be a uniformly convex Banach space, D a bounded closed convex subset of X, and G an open set containing D with $\operatorname{dist}(D, X\setminus G) > 0$ and $0 \in \operatorname{int}(D)$. Suppose $T:\overline{G} \to X$ is a continuous pseudo-contractive mapping which sends bounded sets into bounded sets and which satisfies

(i) $\quad \lambda \in \mathcal{E}_{\partial D} \Longrightarrow \mathcal{E}_D \cap [1, \lambda) \neq \emptyset.$

Then T has a fixed point in D.

Theorem 5. Let X be a uniformly convex Banach space, D a closed convex subset of X with $0 \in \operatorname{int}(D)$ and $T:D \to X$ a continuous pseudo-contractive mapping which sends bounded sets into bounded sets. Suppose that for all $\lambda \in \mathcal{E}_D$ sufficiently near $\lambda_0 = \inf \mathcal{E}_D$ the corresponding eigenvector $\{x_\lambda\}$ is bounded away from ∂D, and moreover suppose that $\liminf_{\lambda \downarrow \lambda_0} \|x_\lambda\| < \infty$. Then T has a fixed point in D.

Proof. Suppose $L = \liminf_{\lambda \downarrow \lambda_0} \|x_\lambda\| < \infty$. If B is a closed ball centered at the origin in X with radius $r > L$, then we may choose a closed neighborhood G of $D \cap B \cap C$ such that $\operatorname{dist}(G, X \setminus D) > 0$, where C is the set $\{x_\lambda\}$. Then the mapping $T:G \to X$ satisfies all the assumptions of Proposition 4 and thus has a fixed point.

We note that if D is a subset of a Banach space X and $T:D \to X$ a pseudo-contractive mapping, then the existence of fixed points is sufficient to guarantee that the set of eigenvectors is bounded. To see this, let x_0 be a fixed point in D and x in D such that $T(x) = \lambda x$ for some $\lambda > 1$. Select $r > 0$ such that $\lambda = (1+r)/r$. By Proposition 1(ii):

$\| x-x_0 \| \leq \| x_0 \|$, and thus $\| x \| \leq 2 \| x_0 \|$.

Remark. In Theorem 5, we do not know whether the Leray-Schauder condition holds on ∂G. However, our weaker boundary condition ($\mathcal{E}$) is clearly fulfilled on ∂G.

Theorem 6. Let X be a uniformly convex Banach space and $T:X \to X$ a continuous pseudo-contractive mapping. If there exists a closed subset D of X with $0 \in \mathrm{int}(D)$ such that

(i) $\lambda \in \mathcal{E}_{\partial D} \implies \mathcal{E}_D \cap [1, \lambda) \neq \emptyset$, and

(ii) $E = \{x \in D : T(x) = \lambda x$ for some $\lambda > 1\}$ is bounded,

then T has a fixed point in X.

First we observe that if T is continuous and pseudo-contractive, then the mappings $tT:X \to X$ for $t \in [0,1)$ are continuous and strongly pseudo-contractive, so such mappings have fixed points by Theorem 6 of Martin [9]. Thus the set

$$\mathcal{E}^+ = \{\lambda > 1 : T(x) = \lambda x \text{ for some } x \in X\}$$

is the interval $(1,\infty)$. Further, $I-tT$ is one-to-one and thus the mapping

$$\psi:(1,\infty) \to X \text{ defined by}$$

$\psi(\lambda) = x_\lambda$ where $Tx_\lambda = \lambda x_\lambda$, is well defined, and by Proposition 1(ii) ψ is continuous.

Proof of Theorem 6. Let $f_r = (1+r)I-rT$ for $r>0$. Then since $I-T$ is accretive, f_r is strongly accretive; hence $f_r(X) = X$ by [9, Theorem 6]. The mapping $g:X \to X$ defined by $g = f_r^{-1}$ is nonexpansive. Now, if there exists a sequence $\{\lambda_n\} \subset \mathcal{E}_D$ with $\lambda_n \to 1$, choose $x_n \in D$ such that $T(x_n) = \lambda_n x_n$ and $y_n = f_r(x_n)$. Then $g(y_n) = \mu_n y_n$ for some sequence $\{\mu_n\}$ with $\mu_n \to 1$. Since $\{y_n\}$ is bounded (by (ii)) and g is nonexpansive, it follows that g has a fixed point which yields a fixed point of T in X. Suppose then, there exists $\lambda_0 > 1$ such that $x_\lambda \notin D$ for all $\lambda \in (1, \lambda_0)$. Let

$$\bar{\lambda} = \sup\{\lambda_0 > 1 : x_\lambda \notin D \text{ for all } \lambda \in (1, \lambda_0)\},$$

and let $\lambda_n > \bar{\lambda}$ for all n such that $\lambda_n \to \bar{\lambda}$ and $x_{\lambda_n} \in D$. By continuity of ψ, $x_{\bar{\lambda}} \in D$ and by Proposition 2, $x_{\bar{\lambda}}$ must be in ∂D. However, by condition (i), there exists $\mu \in \mathscr{C}_D$ such that $\mu < \bar{\lambda}$ and this contradicts the maximality of $\bar{\lambda}$.

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