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NEGATIVE POWERS AND THE SPECTRUM OF MATRICES

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Abstract: A proof is given that for each natural k and each nxn complex valued regular matrix A, we can write

$$A^{-k} = \sum_{i=1}^{n} \nu_{i,-k} A^{i-1},$$

where $\nu_{i,k}$ may be expressed by rational functions $w_{i,-k}$ of the eigenvalues of A. Explicit expressions for $w_{i,-k}$ were found. We have applied these results to obtain estimates for the norms of negative powers of transformations on an n-dimensional normed space with constrained spectrum. These estimates represent considerable strenghtening of results of J. Daniel and T. Palmer.

Key words: Negative powers, norm of iterates.

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1. Introduction. It is a simple matter, via the Cayley-Hamilton theorem, to show that the k-th power for each integer k of an axn matrix A can be represented as a linear combination of the matrices I, A, $A^2, \dots, A^{n-1}$. The coefficients in these combinations are known rational functions of the coefficients appearing in the characteristic equation of A [1, 5, 9, 10]. The last coefficients being elementary symmetric polynomials of the eigenvalues of A, we can write

$$A^k = \sum_{i=1}^{n} \nu_{i,k} A^{i-1}, \tag{1}$$

where $\nu_{i,k}$ may be expressed by rational functions $w_{i,k}$ of the eigenvalues of A. For $k > 0$, $w_{i,k}$ are known polynomials

[4, 7, 11], they proved to be useful in studying the relations between the norm of iterates and the spectral radius [3, 4, 6, 7, 11].

It is the purpose of the present paper to give explicit expressions for $w_{i,k}$ for negative values of k and to apply them to obtain estimates for the norms of negative powers of transformations on an n-dimensional normed space with constrained spectrum.

2. Definitions and preliminaries. Let n be an arbitrary but fixed integer. For i = 1,...,n, we shall define the polynomials

$$E_i = E_i(x_1,\dots,x_n) = \sum_{\substack{e_j \in \{0,1\} \\ e_1+\dots+e_n = i}} x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$$

and

$$a_i = a_i(x_1,\dots,x_n) = (-1)^{n-i} E_{n-i+1}(x_1,\dots,x_n),$$

where $x_1,\dots,x_n$ are considered as indeterminates. Hence

$$(x - x_1)(x - x_2)\dots(x - x_n) = x^n - a_1 - a_2 x - \dots - a_n x^{n-1}.$$

Put

$$b_i(x_1,\dots,x_n) = \begin{cases} 1/a_1 & \text{for } i = n, \\ -a_{i+1}/a_1 & \text{for } i = 1,\dots,n-1. \end{cases}$$

For each i, $1 \le i \le n$, and $k \le n-1$, we shall define rational functions $w_{i,k} = w_{i,k}(x_1,\dots,x_n)$ by the recursive relations

(2) $$w_{i,k} = \sum_{j=1}^{n} b_j w_{i,k+j}$$

with initial conditions

(3) $$w_{i,k}(x_k,\dots,x_n) = \delta_{i,k+1}, \quad 0 \le k \le n-1.$$

To prove that $w_{i,k}$ are the functions spoken about in the introduction, suppose that A is a regular operator on an n-dimensional linear space, and that the eigenvalues of A are $\rho_1, \ldots, \rho_n$. Note that the polynomial

$$f(x) = x^n - \sum_{i=1}^{n} a_i(\rho_1, \ldots, \rho_n) x^{i-1}$$

is the characteristic polynomial of A and that, for $i = 1, \ldots, n$, $w_{i,-1} = b_i$. It is now a simple consequence of the Cayley-Hamilton theorem that

$$(4) \qquad A^{-1} = \sum_{i=1}^{n} b_i(\rho_1, \ldots, \rho_n) A^{i-1},$$

so

$$(5) \qquad A^k = \sum_{i=1}^{n} w_{i,k}(\rho_1, \ldots, \rho_n) A^{i-1}$$

holds for $k = n-1, n-2, \ldots, 0, -1$. To prove (5) for $k < -1$ by induction, suppose that $\ell < -1$ and that (5) is satisfied for $k = \ell + 1, \ell + 2, \ldots, n-1$. Put $\beta_i = b_i(\rho_1, \ldots, \rho_n)$ and $\nu_{i,k} = w_{i,k}(\rho_1, \ldots, \rho_n)$. If we multiply (4) by $A^{\ell+1}$ and use the induction hypothesis, we successively get

$$A^{\ell} = \sum_{i=1}^{n} \beta_i A^{\ell+i} = \sum_{i=1}^{n} \beta_i \sum_{j=1}^{n} \nu_{j,\ell+i} A^{j-1} =$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \beta_i \nu_{j,\ell+i} \right) A^{j-1} = \sum_{j=1}^{n} \nu_{j,\ell} A^{j-1}.$$

For $k \geq n$, the polynomials $w_{i,k}$ may be defined [1, 3, 6] by

$$(6) \qquad w_{i,k+n} = \sum_{j=1}^{n} a_j w_{i,k+j-1}, \quad i = 1, \ldots, n$$

and (3).

3. <u>General expressions</u>. Put

$$T = T(x_1, \dots, x_n) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}$$

and note that

$$T^{-1} = \begin{bmatrix} b_1 & b_2 & \dots & b_{n-1} & b_n \\ 1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

If

$$W_k = \begin{bmatrix} w_{1,k} & w_{2,k} & \dots & w_{n,k} \\ w_{1,k+1} & w_{2,k+1} & \dots & w_{n,k+1} \\ \cdot & \cdot & \dots & \cdot \\ w_{1,k+n-1} & w_{2,k+n-1} & \dots & w_{n,k+n-1} \end{bmatrix}$$

we have by (2) for $k \leq 0$

$$W_{k-1} = T^{-1} W_k$$

and by (6)

$$W_{k+1} = T\, W_k$$

for $k \geq 0$. Since $W_0 = (\delta_{i,j}) = I$, we get

$$W_k = T^k$$

for each integer k.

For $k \geq n$ and $i = 1, \dots, n$, the polynomials $w_{i,k}$ may be expressed [4, 7, 11] by

(7)

$$w_{i,k}(x_1, \dots, x_n) = (-1)^{n-i} \sum_{\substack{e_1 + \dots + e_n = k-i+1 \\ e_i \geq 0}} (q(e_1, \dots, e_n) - 1) x_1^{e_1} \dots x_n^{e_n},$$

where $q(e_1,\dots,e_n)$ denotes the number of e_j different from zero.

We shall use this result to compute the negative powers of T.

Put $D = (\delta_{i,n-i+1})$ and note that $D^{-1} = D$. Simple computations show that

$$(8) \qquad T^{-1}(x_1,\dots,x_n) = DT^k(1/x_1,\dots,1/x_n)D$$

for $k \geq 0$. Comparing the entries in the first row of the matrices in (8), we get

$$(9) \qquad w_{i,-k}(x_1,\dots,x_n) = w_{n-i+1,k+n-1}(1/x_1,\dots,1/x_n)$$

for $i = 1,\dots,n$ and $k \geq 0$.

We have proved the following theorem:

Theorem 1. Let A be a regular operator on an n-dimensional linear space, let the eigenvalues of A be $\rho_1,\dots,\rho_n$ and let $k > 0$. Then

$$(10) \qquad A^{-k} = \sum_{i=1}^{n} w_{i,-k}(\rho_1,\dots,\rho_n)A^{i-1}$$

where

$$(11) \quad w_{i,-k}(\rho_1,\dots,\rho_n) = (-1)^{i-1} \sum_{\substack{e_1+\dots+e_n=k+i-1 \\ e_i \geq 0}} (q(e_1,\dots,e_n)-1) \rho_1^{-e_1}\dots\rho_n^{-e_n}.$$

Note that $w_{i,-k}$ is a polynomial in $1/\rho_1,\dots,1/\rho_n$ and that the sign of all the coefficients in this polynomial depends on i only. For the polynomials $w_{i,k}$, $k \geq n$, this result was known earlier; it was suggested by Professor V. Pták [6] and first proved by the late Professor V. Knichal (unpublished).

4. On $|A^{-k}|$, $|A|$ and $|A^{-1}|_\sigma$. In this section, we shall concern with problems of a nature similar to that raised by J. Daniel and T. Palmer in [2].

Let X_n be an n-dimensional linear space, let $P(X_n)$ be the set of all norms on X_n and let $L(X_n)$ be the algebra of all linear operators on X_n. If $A \in L(X_n)$ and $p \in P(X_n)$, then we shall denote the operator norm of A in the Banach space (X_n, p) by $p(A)$. The spectral radius of $A \in L(X_n)$ will be denoted by $|A|_\sigma$.

Theorem 2. Let $0 < R$, $0 < B$. If $A \in L(X_n)$, $p \in P(X_n)$, $p(A) \leq B$ and $|A^{-1}|_\sigma \leq R$, then for each $k \geq i$

$$p(A^{-k}) \leq \sum_{i=1}^{n} \binom{k+i-2}{i-1} \binom{k+n-1}{n-i} B^{i-1} R^{k+i-1}. \tag{12}$$

Proof: Let R,k,p and A satisfy the assumptions of the theorem and let $\rho_1, \ldots, \rho_n$ be the eigenvalues of A. Since $|A^{-1}|_\sigma = R$, we have $1/|\rho_i| \leq R$. All the coefficients in (11) being of the same sign, we can write

$$\begin{aligned} p(A^{-k}) &= p\left(\sum_{i=1}^{n} w_{i,-k}(\rho_1, \ldots, \rho_n) A^{i-1}\right) \leq \\ &\leq \sum_{i=1}^{n} |w_{i,-k}(1/R, \ldots, 1/R)| B^{i-1} \end{aligned} \tag{13}$$

To finish the proof, it is enough to evaluate $w_{i,-k}(1/R, \ldots$ $\ldots, 1/R)$. This may be done directly or via (9) and results of [4].

In [2], J. Daniel and T. Palmer proved, that for each $B > 0$, there is a number $S_n(B)$ such that $A \in L(X_n)$, $p \in P(X_n)$, $|A^{-1}|_\sigma \leq$ ≤ 1 and $p(A) \leq B$ implies $p(T^{-1}) \leq S_n(B)$. Their result is a special case of the theorem 2. Let us state the quantitative refinement of their result as a corollary:

Corollary 1. Let $B>0$, $A\in L(X_n)$, $p\in P(X_n)$, $|A^{-1}|_\sigma \leq 1$ and $p(A)\leq B$. Then

(14) $$p(A^{-1}) \leq ((B+1)^n - 1)/B.$$

Proof: Put $k = R = 1$ in (12).

Now we are going to show that for small r and $B = 1$, the formula (12) gives the best possible bound.

Denote by $B_{n,\infty}$ the comples n-dimensional vector space, the norm $|x|_\infty$ of the vector $x = (x_1,\dots,x_n)$ being defined by the formula

$$|x|_\infty = \max_{i=1,\dots,n} |x_i| .$$

Regarding a matrix $A = (a_{ij})$ as an operator on $B_{n,\infty}$, we may write

$$|A|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| .$$

Theorem 3. Let $0< r\leq 2^{1/n} - 1$ and $k\geq 1$. Put $\alpha_i = (-1)^{n-i} \binom{n}{n-i+1} r^{n-i+1}$, $i = 1,\dots,n$ and

$$T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot\cdot & \cdot\cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n \end{bmatrix}.$$

Then

$$|T|_\infty = 1, \ |T^{-1}|_\sigma = 1/r$$

and

$$|T^{-k}|_\infty = \max\{|A^{-k}|_\infty : A \ L(B_{n,\infty}), |A|_\infty \leq 1, |A^{-1}|_\sigma \leq 1/r\} = \\ = \sum_{i=1}^{n} \binom{k+i-2}{i-1}\binom{k+n-1}{n-i} / r^{k+i-1} .$$

Proof: Let r and k satisfy the assumptions of the theorem.

If $r \leq 2^{1/n} - 1$, then

$$\sum_{i=1}^{n} |\alpha_i| = \sum_{i=1}^{n} \binom{n}{n-i+1} r^{n-i+1} = (1+r)^n - 1 \leq 1,$$

so that $|T|_\infty = 1$.

Note that the polynomial

$$f(x) = x^n - \sum_{i=1}^{n} \alpha_i x^{n-i} = (x-r)^n$$

is the characteristic polynomial of T. All the roots of the equation $f(\xi) = 0$ being equal to r, we have $|T^{-1}|_\sigma = 1/r$.

Since the first row of the matrix R^{-k} is equal to

$w_{1,-k}(r,\dots,r),\dots,w_{n,-k}(r,\dots,r)$,

we have

$$|T^{-k}|_\infty = \sum_{i=1}^{n} |w_{i,-k}(r,\dots,r)| = \sum_{i=1}^{n} \binom{k+i-2}{i-1}\binom{k+n-1}{n-i} / r^{k+i-1}.$$

The rest follows from the theorem 2.

For special norms it is possible to get far lower bounds. For instance, N.J. Young has proved [12] that for the Hilbert norm $|\cdot|$ and $R>0$,

$$\sup\{|A^{-1}| : A \in L(X_n),\ |A| \leq 1,\ |A^{-1}|_\sigma \leq R\} = R^n,$$

while, by the theorems 2 and 3, for $R \geq (2^{1/n} - 1)^{-1}$

$$\sup\{p(A^{-1}) : p \in P(X_n),\ A \in L(X_n),\ p(A) \leq 1,\ |A^{-1}|_\sigma \leq R\} =$$

$$= \sup\{|A^{-1}|_\infty : A \in L(B_{n,\infty}),\ |A|_\infty \leq 1,\ |A^{-1}|_\sigma \leq R\} =$$

$$= (1+R)^n - 1.$$

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