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A METHOD FOR CONSTRUCTING SOME ENDOMORPHIC UNIVERSES

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Abstract: In this paper there is introduced a method of constructing endomorphic universes satisfying certain conditions dealing with their location in the universal class, for instance endomorphic universes separating two classes.

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Secondary 02H13

We assume the reader to be familiar with the first two chapters of the book [V]. When using results from elsewhere, we recall them.

We are going to study endomorphic universes, introduced in the last chapter of [V]. A class A is an endomorphic universe iff there is an endomorphism F such that F"V = A, the function F being an endomorphism iff its domain equals V and for each set-formula $\varphi(x_1,\dots,x_n)$ of the language FL and each $y_1,\dots,y_n$ the following holds

$$\varphi(y_1,\dots y_n) \equiv \varphi(F(y_1),\dots F(y_n)).$$

Endomorphic universes observed from inside can play the role of the universal class and are variously located in it.

For our method the following equivalent characterization of endomorphic universes, which can be found in the paper [S-V 1], will be essential.

A class A is an endomorphic universe iff the conditions (EUa) and (EUb) are satisfied

(EUa) If $\varphi(z)$ is a set-formula of the language FL_A then we have $(\exists x)\,\varphi(x) \longrightarrow (\exists x \in A)\,\varphi(x)$.

(EUb) For every countable function $F \subseteq A$ there is a set-function f in A prolonging F, i.e. $F \subseteq f$.

We shall concern us with the location of endomorphic universes in the universal class. In the first section we introduce the concept of reserve and we present all that is necessary for our method. We demonstrate this method by constructing an endomorphic universe separating two classes X, Y if the reserve of the former with respect to the later is revealed. In the second and third section we deal with conditions under which the reserves are revealed and we show some applications of the possibility to separate classes by an endomorphic universe. In the end we present two more complicated examples, namely a construction of a non-revealed endomorphic universe intersecting the class γ-FN for each infinite γ and a construction of a monotonous countable sequence of endomorphic universes having Def as its intersection.

Now we recall for convenience some frequently used concepts from the Alternative Set Theory.

A class X is called revealed if for each countable subclass Y there is a set u such that $Y \subseteq u \subseteq X$. Each set-theoretically definable class is revealed. If $\varphi(x,X)$ is a normal formula of the language FL_V then the class $\{z;\ \varphi(z,Z)\}$ is re-

vealed for each set-theoretically definable Z.
An intersection of at most countably many revealed classes is a revealed class. If $\{X_n; n \in FN\}$ is a sequence of non-empty revealed classes such that $X_{n+1} \subseteq X_n$ for each n, then $\bigcap \{X_n; n \in FN\}$ is a revealed and non-empty class.

The set y is said to be definable from the class X iff there is a set-formula $\varphi(z)$ of the language FL_X such that

$$(\exists ! z)\, \varphi(z) \;\&\; \varphi(y).$$

The class of all sets definable from X is denoted Def_X. An ordered pair is definable from its elements and conversely. If $\{M_\alpha ; \alpha \in T\}$ is a sequence of classes, where T is either FN or Ω and $M_\alpha \subseteq M_\beta$ if $\alpha \leq \beta$, then $Def_{\bigcup\{M_\alpha;\alpha\in T\}} = \bigcup\{Def_{M_\alpha}; \alpha \in T\}$.

We suppose that W is a fixed well-ordering of the type Ω of the universal class V. Each proper initial segment with respect to such ordering is countable.

I.

Theorem 1. Let $\varphi(x)$ be a set-formula of the language FL_X. Then the class $\{x; \varphi(x)\}$ is either finite, in which case it is a subclass of Def_X, or it contains at least countably many elements from Def_X.

Proof. Let F be a one-one mapping of N onto V definable by a set-formula of the language FL. The sets x_n satisfying

$$\varphi(x_n) \;\&\; (\forall y)(\varphi(y) \longrightarrow (\exists m<n)(y=x_m) \vee (F^{-1}(y) \geq F^{-1}(x_n)))$$

are definable by a suitable set-formula of the language FL_X. The theorem follows.

Corollary. Let $X = Def_X$. Then X satisfies the condition (EUa).

We define the reserve of X with respect to Y for each X, Y as follows:

$$Rsv\ (X,Y) = \{x;\ Def_{X\cup\{x\}} \cap Y = 0\}.$$

Theorem 2. The following statements hold:

(a) $Rsv\ (X,Y) \cap Y = 0$,

(b) $Y = \bigcup\{Y_\xi;\ \xi \in K\} \longrightarrow Rsv\ (X,Y) = \bigcap\{Rsv\ (X,Y_\xi);\ \xi \in K\}$,

(c) $Def_X \cap Y = 0 \equiv Rsv\ (X,Y) \neq 0 \equiv Rsv\ (X,Y) \supseteq Def_X$,

(d) $X \subseteq X_1\ \&\ Y \subseteq Y_1 \longrightarrow Rsv\ (X_1,Y_1) \subseteq Rsv\ (X,Y)$,

(e) $x \in Rsv\ (X \cup \{z\},Y) \equiv \langle x,z\rangle \in Rsv\ (X,Y)$,

(f) Let $X = \bigcup\{X_n; n \in FN\}$, $Y = \bigcup\{Y_n; n \in FN\}$ with $X_n \subseteq X_{n+1}, Y_n \subseteq Y_{n+1}$ for each $n \in FN$. Then $Rsv\ (X,Y) = \bigcap\{Rsv\ (X_n,Y_n); n \in FN\}$.

Proof. The statements (a)-(e) are trivial. (f) follows from the fact that for each x

$$Def_{X\cup\{x\}} \cap Y = \bigcup\{Def_{X_n \cup\{x\}} \cap Y_n; n \in FN\}.$$

This can be seen by considering de Morgan laws because $m \leq n$ implies

$$Def_{X_n \cup\{x\}} \cap Y_m \subseteq Def_{X_n \cup\{x\}} \cap Y_n.$$

Theorem 3. Let $Rsv\ (X,Y)$ be a revealed class and z an element of $Rsv\ (X,Y)$. Then $Rsv\ (X \cup \{z\},Y)$ is a non-empty revealed class.

Proof. The classes $Def_{X\cup\{z\}}$ and Y are disjoint because z is an element of $Rsv\ (X,Y)$. By the theorem 2c $Rsv\ (X \cup \{z\},Y)$ is non-empty.

By the theorem 2e $\mathrm{Rsv}(X \cup \{z\}, Y) = (\mathrm{Rsv}(X,Y))"\{z\}$. This is a revealed class because $\mathrm{Rsv}(X,Y)$ is revealed.

We note that the part (b) of the following theorem is dispensable in this first section.

<u>Theorem 4</u>. (a) Let $\mathrm{Rsv}(X,Y)$ be a non-empty revealed class, $\alpha \in \Omega$ and H a function defined on $\alpha \cap \Omega$ such that $H(\beta)$ is either 0 or an element of $\mathrm{Rsv}(X \cup H"\beta, Y)$ for $\beta \in \alpha \cap \Omega$. Then the class $\mathrm{Rsv}(X \cup H"\alpha, Y)$ is non-empty and revealed.

(b) Let $\mathrm{Rsv}(X,Y)$ be a non-empty and revealed class and $\mathrm{Rsv}(X,\{y\})$ revealed for each y. Let α be an element of Ω and H, G functions such that $\mathrm{dom}(H) = \alpha \cap \Omega$, $\mathrm{dom}(G) \subseteq \alpha \cap \Omega$, $H(\beta)$ is either 0 or an element of $\mathrm{Rsv}(X \cup H"\beta, Y \cup G"\beta)$ for $\beta \in \alpha \cap \Omega$ and $G(\beta)$ is not an element of the class $\mathrm{Def}_{X \cup H"(\beta+1)}$ whenever G is defined for β.
Then the class $\mathrm{Rsv}(X \cup H"\alpha, Y \cup G"\alpha)$ is non-empty and revealed.

<u>Proof</u>. We proceed by transfinite induction.
(a) For $\gamma = 0$ the class $\mathrm{Rsv}(X \cup H"\gamma, Y)$ equals $\mathrm{Rsv}(X,Y)$ and as such it is non-empty and revealed. Let us assume that it is true for $\beta < \gamma \leq \alpha$.

If γ is a limit number in Ω, let $\{\beta_n; n \in FN\} \subseteq \Omega$ be a sequence such that $\bigcup\{\beta_n \cap \Omega; n \in FN\} = \gamma \cap \Omega$. By the theorem 2f $\mathrm{Rsv}(X \cup H"\gamma, Y)$ is the intersection of the monotonous sequence of non-empty and revealed classes $\mathrm{Rsv}(X \cup H"\beta_n, Y)$ and therefore it is non-empty and revealed, too.

If γ is a successor then the desired property of $\mathrm{Rsv}(X \cup H"\gamma, Y)$ follows from the inductive assumption and the theorem 3 because $H(\gamma - 1)$ is either 0 or an element of

Rsv $(X \cup H''(\gamma -1), Y)$.

(b) The steps for 0 and a limit number γ are the same as above. Let γ be a successor and Rsv $(X \cup H''(\gamma -1), Y \cup G''(\gamma -1))$ a non-empty and revealed class. The same is true about Rsv $(X \cup H''\gamma, Y \cup G''(\gamma -1))$ by the theorem 3. If G is not defined for $\gamma - 1$ then the proof is complete.

In the other case $G(\gamma -1)$ is not an element of $\mathrm{Def}_{X \cup H''\gamma}$. The theorem 2c implies non-emptiness of the class Rsv $(X, \{G(\gamma -1)\})$ which is moreover by our assumption revealed. Obviously, $H(\beta)$ is an element of Rsv $(X \cup H''\beta, \{G(\gamma -1)\})$ for each $\beta \in \gamma \cap \Omega$. Thus the class Rsv $(X \cup H''\gamma, \{G(\gamma -1)\})$ is non-empty and revealed by (a). By the theorem 2b the class Rsv $(X \cup H''\gamma, Y \cup G''\gamma)$ is the intersection of the reserves of $X \cup H''\gamma$ with respect to $Y \cup G''(\gamma -1)$ and $\{G(\gamma -1)\}$. Consequently, it is non-empty and revealed. (Each non-empty reserve contains Def.)

Let $\{F_\alpha ; \alpha \in \Omega\}$ be a fixed sequence of at most countable functions such that each such function $F \subseteq V$ occurs here uncountably many times. Actually, there is no such object in the extended universe, but we can imagine that we are working with a suitable coding pair because the system of all countable classes is codable. In the same sense we use sequences of classes elsewhere, too.

Theorem 5. Let $\{M_\alpha ; \alpha \in \Omega\}$ be a sequence of classes, $M_\alpha = \mathrm{Def}_{M_\alpha}$ for each α and $M_\alpha \subseteq M_\beta$ if $\alpha \leq \beta$. Suppose that for each α the following holds: if F_α is a subclass of M_α then F_α has a prolongation in $M_{\alpha+1}$, i.e. there is a set f in $M_{\alpha+1}$ such that $f \supseteq F_\alpha$.

Then the class $M = \bigcup \{ M_\alpha ; \alpha \in \Omega \}$ is an endomorphic universe.

Proof. By the corollary of the first theorem M satisfies the condition (EUa), because $M = Def_M$. Let $F \subseteq M$ be a countable function. Obviously, there is $\alpha \in \Omega$ such that $F \subseteq M_\alpha$ and $\beta \geq \alpha$ with $F = F_\beta$. Consequently, F has a prolongation in $M_{\beta+1} \subseteq M$. We have proved the condition (EUb).

Theorem 6. Let Rsv (X, Y) be a revealed and non-empty class. Then there is an endomorphic universe A such that $X \subseteq A$ and $Y \cap A = 0$.

Proof. We shall define a function H for $\alpha \in \Omega$ such that $H(\alpha)$ is the first element (w.r.t. W) in the class Rsv $(X \cup H''\alpha, Y)$ prolonging F_α if F_α is a subclass of $Def_{X \cup H''\alpha}$ and $H(\alpha) = 0$ in the other case.

It follows immediately from the theorems 2c and 4a that we can proceed by transfinite recursion, because the classes Rsv $(X \cup H''\alpha, Y)$ remain revealed and contain $Def_{X \cup H''\alpha}$.

The class $A = Def_{X \cup H''\Omega}$ is an endomorphic universe by the theorem 5 (we set $M_\alpha = Def_{X \cup H''\alpha}$). By the theorem 2a the classes A and Y are disjoint, because A is constructed to be a subclass of Rsv (X,Y).

II. This section is devoted to some important properties of the classes Rsv (X,Y) and Def_X. We shall use a lemma describing the structure of these classes. The lemma is rather technical and requires the following notation.

For each set-formula $\varphi(y, x_1, x_2, \ldots x_n)$ of the language FL let

$$T_\varphi = \{ \langle y, x_1, \ldots x_n \rangle ; \varphi(y, x_1, \ldots x_n) \& (\forall z)(\varphi(z, x_1, \ldots x_n) \rightarrow z = y) \}.$$

Let us recall a property of ordered n-tuples:

$$\langle y,x_1,x_2,\dots x_n\rangle = \langle y, \langle x_1,x_2,\dots x_n\rangle\rangle .$$

Note that each class T_φ is a function and a set-theoretically definable class and that the class Φ of all set-formulas of the language FL is a most countable.

<u>Lemma</u>. For each X, Y we have the following

(a) $\mathrm{Def}_X = \bigcup\{ T_\varphi "X^n;\ \varphi(y,x_1,\dots x_n)\in\Phi\}$,

(b) $\mathrm{Rev}\,(X,Y) = V - \bigcup\{(T_\varphi^{-1}"Y)"X^{n-1};\ \varphi(y,x_1,\dots x_n)\in\Phi\} =$
$= \bigcap\{ V - (T_\varphi^{-1}"Y)"X^{n-1};\ \varphi(y,x_1,\dots x_n)\in\Phi\}$.

<u>Proof</u>. Both the assertions can be seen from the fact that a set z is an element of Def_Z iff there is a set-formula $\varphi(y,x_1,\dots x_n)$ of the language FL and $z_1,z_2,\dots z_n$ in Z such that

$$\varphi(z,z_1,\dots z_n)\ \&\ (\forall y)(\varphi(y,z_1,\dots z_n)\longrightarrow y=z).$$

<u>Theorem 7</u>. Let u be a set, α the number of its elements and $\alpha\in$ N-FN. Then for each γ in N-FN there is a set v set-subvalent to α^γ and containing Def_u as a subclass.

<u>Proof</u>. By the previous lemma Def_u is the union of countably many sets $T_\varphi "u^n$, each set-subvalent to α^n for some $n\in$ FN.

The prolongation axiom implies the existence of a set w set-subvalent to α and such that each $T_\varphi "u^n$ is an element of w and for each $x\in w$ is $x \precsim \alpha^{\gamma-1}$. Setting $v = \bigcup w$, we obtain a set with the desired property.

<u>Theorem 8</u>. (a) Let X, Y be set-theoretically definable classes or - more generally - let the pair $\langle X,Y\rangle$ be fully

revealed. [See [S-V 1].) Then the class Rsv (X,Y) is revealed.

(b) Let X, Y be σ-classes. Then the class Rsv (X,Y) is revealed.

<u>Proof</u>. By the lemma the class Rsv (X,Y) is an intersection of countably many classes, each definable by a normal formula of the language FL with the only class parameters X,Y. Under the assumption made in (a) such classes are revealed. Consequently, Rsv (X,Y) is revealed. The assertion (b) follows from (a) and the theorem 2f.

<u>Corollary</u>. Let X satisfy any of the following conditions:
(a) X is a set-theoretically definable class (more generally: X is fully revealed),
(b) X is a σ-class.
Then the class Rsv $(X,\{y\})$ is revealed for each y.

III. Now we shall concern us with applications of the theorems 6, 7 and 8.

<u>Theorem 9</u>. Let Rsv $(X,\{y\})$ be a revealed class for each set y. Then the intersection of all endomorphic universes containing X is the class Def_X.

<u>Proof</u>. Each endomorphic universe containing X contains Def_X by the condition (EUa). For each $y \notin \mathrm{Def}_X$ there is an endomorphic universe containing X and not $\{y\}$ by the theorem 6.

<u>Theorem 10</u>. (a) Let $w \notin \mathrm{Def}_w$. Then there is an endomorphic universe A such that $w \subseteq A$ and $w \notin A$.

(b) Each infinite set u has a subset w such that $w \notin \mathrm{Def}_w$.

<u>Proof</u>. The first assertion is an easy consequence of

the theorems 6 and 8.

Let α be the number of elements of the set u. The power-set of u, $\mathcal{P}(u)$, has 2^{α} elements. There is an infinite β such that $2^{\alpha} > \alpha^{\beta}$, because the class $\{\beta ; 2^{\alpha} > \alpha^{\beta}\}$ is set-theoretically definable and each finite number belongs to it. Suppose that each subset w of u is definable from its elements: $w \in Def_w \subseteq Def_u$. Then $\mathcal{P}(u) \subseteq Def_u$. By the theorem 7 there is a set v set-subvalent to α^{β} such that $Def_u \subseteq v$. But $\mathcal{P}(u)$ cannot be a subset of v because of the number of its elements.

Theorem 11. There is an endomorphic universe A such that the initial segment R_A,

$$R_A = \{\alpha \in N ; (\forall \beta \leq \alpha)(\beta \in A)\}$$

with the usual addition and multiplication is not a model of PA (= Peano axioms).

Proof. If an initial segment R is a model of PA then it is closed under the operation LM

$$LM(\alpha) = \min \{\beta ; \beta \neq 0 \,\&\, (\forall \gamma)(\gamma \leq \alpha \longrightarrow \gamma \mid \beta)\}.$$

This is easily verified from the fact that the ordering and divisibility in R coincide with the restriction of the corresponding relations in N and that the following formula is provable in PA

$$(\forall \alpha)(\exists \beta)(\beta \neq 0 \,\&\, (\forall \gamma)(\gamma \leq \alpha \longrightarrow \gamma \mid \beta)).$$

Let α be an infinite number. We shall show that there is an infinite β with $LM(\alpha) > \alpha^{\beta}$. For each n, k from FN $LM(\alpha)$ is divisible by $(\alpha - n)$ and $(\alpha - k)$. The greatest common divisor of these numbers is less or equal to $|n-k|$. It follows that for each n

$$\mathrm{LM}(\alpha) \geq \alpha \cdot \frac{\alpha-1}{1} \cdot \frac{\alpha-2}{1\cdot 2} \quad \ldots \cdot \frac{\alpha-n}{n!} \geq \left(\frac{\alpha}{2}\right)^{n+1} \cdot \left(\frac{1}{n!}\right)^{n+1} > \alpha^{n}.$$

The class $\{\beta ; \mathrm{LM}(\alpha) > \alpha^{\beta}\}$ is set-theoretically definable, each finite number belongs to it and consequently an infinite number, too.

Let α, β be infinite numbers such that $\mathrm{LM}(\alpha) > \alpha^{\beta}$. By the theorem 7 there is a set v such that $\mathrm{Def}_{\alpha} \subseteq v$ and v is set-subvalent to α^{β}. Thus we can find a number δ in $\mathrm{LM}(\alpha) - \mathrm{Def}_{\alpha}$.

By the theorems 2c, 8 and 6 there is an endomorphic universe A such that $\alpha \subseteq A$ and $\delta \notin A$. Clearly, R_A is not closed under the operation LM and so it is not a model of PA.

IV.

<u>Theorem 12</u>. There is an endomorphic universe A such that $(\forall \gamma \in N\text{-}FN)(\exists \alpha_1, \alpha_2 \in \gamma\text{-}FN)(\alpha_1 \in A \;\&\; \alpha_2 \notin A)$.

<u>Proof</u>. Let S be a one-one mapping of Ω onto N-FN. We shall define functions H and G for $\alpha \in \Omega$, $H(\alpha) = \langle H_1(\alpha), H_2(\alpha)\rangle$, such that $H_1(\alpha)$ is the first element (with respect to W) in the class $\mathrm{Rsv}\,(H''\alpha, G''\alpha)$ prolonging the function F_{α} if $F_{\alpha} \subseteq \mathrm{Def}_{H''\alpha}$ and $H_1(\alpha)=0$ in the other case: $H_2(\alpha)$ is the first element (with respect to W) in the intersection of $\mathrm{Rsv}\,(H''\alpha \cup \{H_1(\alpha)\}, G''\alpha)$ and $S(\alpha)-\mathrm{FN}$ (it follows by the theorem 3 that $H(\alpha)$ is an element of $\mathrm{Rsv}\,(H''\alpha, G''\alpha)$); $G(\alpha)$ is the first element (with respect to W) in the class $S(\alpha) - \mathrm{FN} - \mathrm{Def}_{H''(\alpha+1)}$.

We can proceed by transfinite recursion, because by the theorem 4b and 2c the classes $\mathrm{Rsv}\,(H''\alpha, G''\alpha)$ remain revealed and contain $\mathrm{Def}_{H''\alpha}$; by the theorem 3 and 2c also the classes $\mathrm{Rsv}\,(H''\alpha \cup \{H_1(\alpha)\}, G''\alpha)$ remain revealed and contain FN and the class $S(\alpha) - FN - \mathrm{Def}_{H''(\alpha+1)}$ is always non-empty, because it is a complement of a countable class to an uncountable one.

Let $A = \mathrm{Def}_{H''\Omega}$. Then A is an endomorphic universe by the theorem 5 (we set $M_\alpha = \mathrm{Def}_{H''\alpha}$). The classes $\mathrm{Rsv}\,(H''\alpha, G''\alpha)$ are non-empty for $\alpha \in \Omega$ and so $A \cap G''\Omega = 0$. Moreover $A \supseteq H_2''\Omega$.

Clearly, A has the desired property.

Theorem 13. There is a countable sequence of endomorphic universes $\{A_n; n \in FN\}$ such that $A_n \supseteq A_{n+1}$ for each n and $\bigcap \{A_n; n \in FN\} = \mathrm{Def}$.

Proof. We shall use the following notation.

If H_n are functions defined on Ω, let

$$A(\alpha,n,k) = \mathrm{Def}_{\bigcup\{H_j''\alpha;\, j \geq k\} \cup \{H_j(\alpha);\, n > j \geq k\}}$$

$$R(\alpha,n) = \bigcap\{\mathrm{Rsv}(A(\alpha,n,k+1),\ A(\alpha,n,k) - A(\alpha,n,k+1);\ k < n\}$$

($R(\alpha,0)$ equals V.)

Let $\prec$ be the lexicographic ordering of the class $\Omega \times FN$,

$$\langle \beta, j \rangle \prec \langle \alpha, n \rangle \equiv \text{either } \beta < \alpha \text{ or } (\beta = \alpha \text{ and } j < n).$$

Note that it is a well-ordering.

Each class $R(\alpha,n)$ is an intersection of finitely many reserves of countable classes and consequently by the theorem 8 a revealed class. Moreover by the theorem 2c each class $R(\alpha,n)$ contains $A(\alpha,n,n)$, because for each k we have $A(\alpha,n,k+1) =$

$= \mathrm{Def}_{A(\alpha,n,k+1)}$ and for $k<n$ $A(\alpha,n,k+1)$ contains the class $A(\alpha,n,n)$.

It follows that we can define by transfinite recursion a function H on $\Omega \times FN$ such that for functions $H_n : H_n(\alpha) = H(\alpha,n)$ the following holds: $H_n(\alpha)$ is the first element (with respect to W) in the class $R(\alpha,n)$ prolonging F_α if $F_\alpha \subseteq A(\alpha,n,n)$ and $H_n(\alpha) = 0$ in the other case.

Let $A_n = \bigcup\{A(\alpha,n,n);\ \alpha \in \Omega\}$. By the theorem 5 each class A_n is an endomorphic universe (we set $M_\alpha = A(\alpha,n,n)$). Obviously $A_{n+1} \subseteq A_n$ for each n.

Let $u \in A_0 - \mathrm{Def}$ and $\langle \alpha_0, n_0 \rangle$ the first $\langle \alpha, n \rangle$ in $\Omega \times FN$ such that u is an element of $A(\alpha,n,0)$. Clearly $\langle \alpha_0, n_0 \rangle \neq \langle 0,0 \rangle$ and $n_0 \neq 0$, because for each k we have

(1) $\alpha \neq 0 \longrightarrow A(\alpha,0,k) = \bigcup\{A(\beta,j,k);\ \langle \beta,j \rangle \prec \langle \alpha,0 \rangle\}$.

Consequently, u is not an element of the class $A(\alpha_0,n_0,n_0)$ because

$$A(\alpha_0,n_0,n_0) = A(\alpha_0,0,n_0) \subseteq A(\alpha_0,0,0)$$

and so there is k_0 such that $u \in A(\alpha_0,n_0,k_0) - A(\alpha_0,n_0,k_0+1)$.

We shall prove by transfinite induction that u is not an element of $A(\alpha,n,k_0+1)$ for each $\langle \alpha,n \rangle$. Then u is not an element of A_{k_0+1} and our assertion concerning the intersection of $\{A_n; n \in FN\}$ follows.

For $\langle \alpha,n \rangle \preceq \langle \alpha_0,n_0 \rangle$ it is true, because for each k we have

(2) $\langle \alpha,n \rangle \prec \langle \beta,j \rangle \longrightarrow A(\alpha,n,k) \subseteq A(\beta,j,k)$.

Let $\langle \alpha,n \rangle \succ \langle \alpha_0,n_0 \rangle$ and assume that u is not an element of $A(\beta,j,k_0+1)$ if $\langle \beta, j\rangle \prec \langle \alpha, n\rangle$.

By (1) u is not an element of $A(\alpha,n,k_0+1)$ if n=0.

If $n \neq 0$ then u is an element of $A(\alpha,n-1,k_0) - A(\alpha,n-1,k_0+1)$ by (2) and the inductive assumption. Either $(n \leq k_0+1)$ or $(H_{n-1}(\alpha) = 0)$ implies that $A(\alpha,n,k_0+1) = A(\alpha,n-1,k_0+1)$. For $(n-1 > k_0)$ and $(H_{n-1}(\alpha) \neq 0)$ we have

$$H_{n-1}(\alpha) \in \mathrm{Rsv}\ (A(\alpha,n-1,k_0+1),\ A(\alpha,n-1,k_0) - A(\alpha,n-1,k_0+1))$$

and by the definition of reserves

$$A(\alpha,n,k_0+1) \cap (A(\alpha,n-1,k_0) - A(\alpha,n-1,k_0+1)) = 0.$$

In either case u is not an element of the class $A(\alpha,n,k_0+1)$.

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