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THE LATTICE OF INDISCERNIBILITY EQUIVALENCES

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Abstract: In this paper we prove that the class of all indiscernibility equivalences creates a lattice in alternative set theory. Every subclass of this lattice has supremum; every countable subclass of it has infimum. Descending sequences ordered by type Ω have no infimum.

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Indiscernibility equivalences (see [V]) play a very important role when we are building mathematics in alternative set theory. The class of all indiscernibility equivalences is ordered by inclusion. In this article it is shown that this class creates a lattice and some properties of this lattice are described.

The whole paper can be considered as immediate continuation of the book [V]. When referring to that book we shall cite only the section and the chapter in question.

In the § 1 Ch. III. the notion of an indiscernibility equivalence is defined. The fact that the intersection of countably many indiscernibility equivalences is an indiscernibility equivalence is also proved in the mentioned chapter.

Every indiscernibility equivalence is a π-class and that is the reason why the class of all indiscernibility equivalences is codable.

In this paper we shall deal with various indiscernibility equivalences. Thus we shall denote e.g. the figure of the class X in the equivalences $\overset{*}{=}$, $\doteq$, etc. by $Fig^*(X)$, $Fig^+(X)$, etc. respectively. We also use the notation $Mon^*(x)$, $Mon^+(x)$ etc.

We use the notation $Fig^+(X)$, $Mon^+(x)$ also in the case if the equivalence $\overset{+}{=}$ is not an indiscernibility equivalence. We use this notation in the following natural sense

$Mon^+(x) = \{y; y \overset{+}{=} x\}$, $Fig^+(X) = \{y; (\exists x \in X)(y \overset{+}{=} x)\}$.

We say that a codable class $\mathfrak{M}$ is a closed base for the indiscernibility equivalence $\overset{+}{=}$ if $\mathfrak{M}$ has the following properties:

(1) $\mathfrak{M}$ is at most countable.

(2) Every $X \in \mathfrak{M}$ is closed in $\overset{+}{=}$.

(3) For every $X, Y \in \mathfrak{M}$, $X \cap Y \in \mathfrak{M}$.

(4) If X is closed in $\overset{+}{=}$ then $X = \bigcap\{Y; X \subseteq Y \in \mathfrak{M}\}$.

<u>Theorem</u>. If $\overset{+}{=}$ is an indiscernibility equivalence then there is a $\mathfrak{M}$ which is a closed base for $\overset{+}{=}$.

<u>Proof</u>: In the § 2 Ch. III it is proved that there is a class Z which is at most countable and such that for every closed figure X the following holds: $X = \bigcap\{Fig^+(u); X \subseteq Fig^+(u) \,\&\, u \in Z\}$. We put $\mathfrak{M}_0 = \{Fig^+(u); u \in Z\}$. Now it is sufficient to put $\mathfrak{M}$ to be the least class such that $\mathfrak{M}_0 \subseteq \mathfrak{M}$ and such that for $Y, X \in \mathfrak{M}$ the formula $X \cap Y \in \mathfrak{M}$ holds.

Theorem. Let $\overset{*}{=}$ be an indiscernibility equivalence. Let $\mathcal{M}$ be a closed base for $\overset{*}{=}$. For every disjoint closed figures X_1, X_2 (in $\overset{*}{=}$) there are $Y_1, Y_2 \in \mathcal{M}$ such that the formulas $X_1 \subseteq Y_1$, $X_2 \subseteq Y_2$, $X_1 \cap Y_2 = \emptyset$, $X_2 \cap Y_1 = \emptyset$, $Y_1 \cup Y_2 = V$ hold.

Proof: X_1, X_2 are disjoint π-classes. Thus there is a set-theoretically definable class Z such that $X_1 \subseteq Z$, $Z \cap X_2 = \emptyset$ hold. Evidently $\mathrm{Fig}^*(Z) \cap X_2 = \emptyset$. Let us put $\mathcal{M}_0 = \{Y;\ \mathrm{Fig}^*(Z) \subseteq \subseteq Y \in \mathcal{M}\}$. We have $\mathrm{Fig}^*(Z) = \bigcap \mathcal{M}_0$. If for every $Y \in \mathcal{M}_0$ the formula $Y \cap X_2 \neq \emptyset$ holds, then $\bigcap \mathcal{M}_0 \cap X_2 \neq \emptyset$ - a contradiction. Thus there is an $Y_1 \in \mathcal{M}_0$ such that $\mathrm{Fig}^*(Z) \subseteq Y_1$ & & $Y_1 \cap X_2 = \emptyset$ holds. The proof of the existence of the class $Y_2 \in \mathcal{M}$ having the properties $\mathrm{Fig}^*(V-Z) \subseteq Y_2$, $Y_2 \cap X_1 = \emptyset$ is analogous. Evidently $X_1 \subseteq Z \subseteq \mathrm{Fig}^*(Z) \subseteq Y_1$ and $X_2 \subseteq Y_2$ hold. We have $V = Z \cup (V-Z) \subseteq \mathrm{Fig}^*(Z) \cup \mathrm{Fig}^*(V-Z) \subseteq Y_1 \cup Y_2$.

Remember that if for any x, y the formula $x \overset{*}{=} y \Rightarrow x \overset{+}{=} y$ holds then the equivalence $\overset{*}{=}$ is called finer than $\overset{+}{=}$.

Theorem. Let $\overset{*}{=}$ be an indiscernibility equivalence. Let $\overset{*}{=}$ be finer than $\overset{+}{=}$. If for any u the class $\mathrm{Fig}^+(u)$ is a π-class then $\overset{+}{=}$ is an indiscernibility equivalence.

Proof: $\overset{+}{=}$ is compact because it is coarser than $\overset{*}{=}$. Let $\mathcal{M}$ be a closed base for $\overset{*}{=}$. Let $X_1, X_2 \in \mathcal{M}$ be such that $X_1 \cup \cup X_2 = V$. We put $R = (\mathrm{Fig}^+(X_1))^2 \cup (\mathrm{Fig}^+(X_2))^2$. As $X_1 \in \mathcal{M}$ there is a u such that $X_1 = \mathrm{Fig}^*(u)$. Thus $\mathrm{Fig}^+(X_1) = \mathrm{Fig}^+(u)$ and $\mathrm{Fig}^+(X_1)$, $\mathrm{Fig}^+(X_2)$ are π-classes. It follows that R is a π-class. As there are at most countably many pairs $X_1, X_2 \in \in \mathcal{M}$, there are at most countably many relations R having the mentioned properties. We prove that the equivalence $\overset{+}{=}$ is the intersection of all such relations R and thus we prove that $\doteq$ is a π-equivalence. Let $x \doteq y$. If $x \in X_1$ then $y \in$ $\in Fig^+(X_1)$, thus $\langle x,y\rangle \in R$. For $x \in X_2$ the proof is analogous. To prove the opposite implication, we suppose $Mon^+(x) \cap Mon^+(y) = \emptyset$. Using the previous theorem we obtain $X_1, X_2 \in \mathfrak{M}$ having the following properties $X_1 \cup X_2 = V$, $Mon^+(x) \subseteq X_1$, $Mon^+(y) \subseteq X_2$, $X_1 \cap Mon^+(y) = \emptyset$, $X_2 \cap Mon^+(x) = \emptyset$. Thus we have $x \notin Fig^+(X_2)$, $y \notin Fig^+(X_1)$ and $\langle x,y\rangle \notin (Fig^+(X_1))^2 \cup (Fig^+(X_2))^2$.

Theorem. Let $\stackrel{*}{=}$ be an indiscernibility equivalence. Let $\mathfrak{M}$ be a closed base for $\stackrel{*}{=}$. If $\stackrel{*}{=}$ is finer than $\doteq$ and if for every x the class $Mon^+(x)$ is a π-class then the following holds

$$x \doteq y \equiv (\forall X \in \mathfrak{M})(x \in Fig^+(X) \equiv y \in Fig^+(X)).$$

Proof: The implication $\Longrightarrow$ is evident. To prove an opposite implication let us suppose $Mon^+(x) \cap Mon^+(y) = \emptyset$. The classes $Mon^+(x)$, $Mon^+(y)$ are figures in the equivalence $\stackrel{*}{=}$ and are also π-classes. Thus there is $X \in \mathfrak{M}$ such that $Mon^+(x) \subseteq X$, $X \cap Mon^+(y) = \emptyset$. It follows that $x \in Fig^+(X)$ and $y \notin Fig^+(X)$.

Theorem. Let $\{=_\alpha ; \alpha \in \Omega\}$ be a class of indiscernibility equivalences. If there is an indiscernibility equivalence $\doteq$ such that $\doteq$ is finer than $=_\alpha$ for every α then there is $\gamma \in \Omega$ such that $\bigcap\{=_\alpha ; \alpha \in \Omega\} = \bigcap\{=_\alpha ; \alpha \in \gamma \cap \Omega\}$. In this case $\bigcap\{=_\alpha ; \alpha \in \Omega\}$ is an indiscernibility equivalence.

Proof: We can suppose that $\alpha < \beta \Longrightarrow =_\beta$ is finer than $=_\alpha$. (Remember that the intersection of at most countably many indiscernibility equivalences is an indiscernibility equivalence.) Let $\mathfrak{M}$ be a closed base for $\doteq$. Let us choose $X \in \mathfrak{M}$, then $\{Fig_{\alpha}(X); \alpha \in \Omega\}$ is a descending sequence of closed figures in the equivalence $\doteq$ and thus using § 2 Ch. III, we obtain a $\gamma \in \Omega$ such that for $\gamma < \alpha \in \Omega$ the formula $Fig_{\alpha}(X) = Fig_{\gamma}(X)$ holds. Because $\mathfrak{M}$ is at most countable, we can suppose that there is a $\gamma \in \Omega$ such that for every $X \in \mathfrak{M}$ and for every $\gamma < \alpha \in \Omega$ the formula $Fig_{\alpha}(X) = Fig_{\gamma}(X)$ holds. Using the previous theorem for $\gamma < \alpha \in \Omega$ we obtain that $Mon_{\alpha}(x) = \bigcap \{Fig_{\alpha}(X); X \in \mathfrak{M} \ \& \ x \in Fig_{\alpha}(X)\} = \bigcap \{Fig_{\gamma}(X); X \in \mathfrak{M} \ \& \ x \in Fig_{\gamma}(X)\} = Mon_{\gamma}(x)$. Thus $=_{\alpha}$, $=_{\gamma}$ are identical. We have proved $\bigcap \{=_{\alpha}; \alpha \in \Omega\} = =_{\gamma}$.

Theorem. For a class $\{=_{\alpha}; \alpha \in \Omega\}$ of indiscernibility equivalences there is an indiscernibility equivalence $\doteq$ having the following properties.

(a) For every $\alpha \in \Omega$ we have $=_{\alpha}$ is finer than $\doteq$.

(b) If $\stackrel{*}{=}$ is an indiscernibility equivalence such that for every $\alpha \in \Omega$ the equivalence $=_{\alpha}$ is finer than $\stackrel{*}{=}$ then $\doteq$ is finer than $\stackrel{*}{=}$.

Proof: Let $\mathfrak{M}$ be the class of all indiscernibility equivalences $\stackrel{*}{=}$ such that for every α, the equivalence $=_{\alpha}$ is finer than $\stackrel{*}{=}$. ($\mathfrak{M}$ is codable as it is a subclass of the class of all π-classes and $V^2 \in \mathfrak{M}$.) $\bigcap \mathfrak{M}$ is an indiscernibility equivalence. (To prove it we use the previous theorem.) $\bigcap \mathfrak{M}$ has evidently the properties required in the theorem.

In the last part of the paper we suppose the reader to be familiar with § 1 Ch. V.

__Theorem.__ Let F be an automorphism. If R is an indiscernibility equivalence then F " R is an indiscernibility equivalence, too.

__Proof:__ F " R is evidently an equivalence. The image of a π-class is a π-class because the image of a set-theoretically definable class is a set-theoretically definable class. Now it suffices to prove the compactness of F " R. Let u be an infinite set, put $v=F^{-1}$"u. v is obviously an infinite set. Using the compactness of R we obtain $x,y \in v$ such that $x \neq y$ and $\langle x,y \rangle \in R$. Thus $F(x)$, $F(y) \in u$, $F(x) \neq F(y)$, $\langle F(x), F(x) \rangle = F(\langle x,y \rangle) \in F$ " R.

Let us put $x \overset{\circ}{=}_X y$ iff for any set-formula $\varphi(z)$ of the language FL_X the formula $\varphi(x) \equiv \varphi(y)$ holds.

The set y is said to be definable using parameters from the class X iff there is a set-formula $\varphi(z)$ of the language FL_X such that the formulas $(\exists ! z)\varphi(z)$ and $\varphi(y)$ hold. We use the notation Def_X for the class of all sets definable using parameters from the class X.

If a class X is countable then for the class Def_X and for the equivalence $\overset{\circ}{=}_X$ the analogues of assertions concerning the class Def and the equivalence $\overset{\circ}{=}$ hold. Especially $\overset{\circ}{=}_X$ is a totally disconnected indiscernibility equivalence.

__Theorem:__

(a) $X \in \mathrm{Def}_X$.

(b) $Y \subseteq \mathrm{Def}_X \equiv \mathrm{Def}_Y \subseteq \mathrm{Def}_X$.

(c) For any set-formula of the language FL_{Def_X} there is an equivalent set-formula of the language FL_X.

(d) The equivalence $\overset{\circ}{=}_X$ is finer than $\overset{\circ}{=}_Y$ iff $\mathrm{Def}_Y \subseteq \mathrm{Def}_X$.

(e) Let F be an automorphism. The image of the equivalence $\overset{\circ}{\underset{X}{=}}$ is the equivalence ${}_{F}\overset{\circ}{\underset{"X}{=}}$ and F " $\mathrm{Def}_X = \mathrm{Def}_{F"X}$ holds.

Theorem. For any class X which is at most countable there is a set u such that the formula $X \subseteq \mathrm{Def}_{\{u\}}$ holds.

Proof: Let G be a function such that $X = G$ "FN. Let g be a prolongation of G. We prove $X \subseteq \mathrm{Def}_{\{g\}}$. Let $x \in X$, let $n \in FN$ be such that $x = G(n)$. Thus $x = g(n)$. We have $g \in \mathrm{Def}_{\{g\}}$ and $n \in \mathrm{Def}_{\{g\}}$. Thus $x \in \mathrm{Def}_{\{g\}}$.

Theorem. For any indiscernibility equivalence $\overset{+}{=}$ there is a set u such that ${}_{\{u\}}\overset{\circ}{=}$ is finer than $\overset{+}{=}$.

Proof. Let $\{R_n; n \in FN\}$ be a generating sequence for $\overset{+}{=}$. As R_n are set-theoretically definable classes, there is a class C which is at most countable and a sequence $\{\varphi_n(x,y); n \in FN\}$ of set-formulas of the language FL_C such that $R_n = \{\langle x,y\rangle ; \varphi_n(x,y)\}$. Let $\{v_n; n \in FN\}$ be a sequence such that v_n is a maximal R_n-net on V. For every n the set v_n is finite. Using the previous theorem we obtain a set u such that $C \cup \{v_n; n \in FN\} \subseteq \mathrm{Def}_{\{u\}}$. We prove that ${}_{\{u\}}\overset{\circ}{=}$ is finer than $\overset{+}{=}$. Let $x \overset{\circ}{\underset{u}{=}} y$ and $\neg\, x \overset{+}{=} y$ hold. In this case there is an n such that $\langle x,y\rangle \notin R_n$. Thus there is $z \in v_{n+1}$ such that $\langle x,z\rangle \in R_{n+1}$, $\langle y,z\rangle \notin R_{n+1}$. Thus $\varphi_{n+1}(x,z)$ and $\neg\, \varphi_{n+1}(y,z)$. But the formula $\varphi_{n+1}(x,z)$ is equivalent with a set formula $\psi(x)$ of the language $FL_{\{u\}}$. Thus $\neg\, x\, {}_{\{u\}}\overset{\circ}{=} y$ - a contradiction.

The following theorem is a special case of the previous one.

Theorem. For any indiscernibility equivalence $\overset{+}{=}$ there is an indiscernibility equivalence $\overset{\#}{=}$ having the following properties.

(a) $\overset{*}{=}$ is finer than $\overset{+}{=}$.

(b) $\overset{*}{=}$ is totally disconnected.

(c) Every clopen monad in the equivalence $\overset{*}{=}$ is a singleton.

R e f e r e n c e

[V] P. VOPĚNKA: Mathematics in the Alternative Set Theory, Teubner-Texte zur Mathematik, Leipzig 1979.

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