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LEFT-SEPARATED SPACES: A COMMENT TO A PAPER OF M. G. TKAČENKO

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Abstract: There appeared two beautiful papers of M.G. Tkačenko [T_1][T_2] in the last issue of this journal. He studied the properties of spaces which can be expressed as a union of not too many left-separated subspaces. In this note we want to give alternative (and perhaps easier) proofs of Tkačenko's theorems.

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0. Preliminaries. A topological space X is called left-separated (right-separated, resp.), if there exists a well-ordering $<$ of a set X such that each initial (coinitial, resp.) segment under $<$ is closed. It turns out that left-separated spaces have other pleasant properties, cf. e.g. [A_1],[A_2],[GJ]. Gerlitz and Juhász ([GJ]) proved among others, that each left-separated compact Hausdorff space X is both scattered and sequential, Tkačenko ([T_2]) showed that the same holds if the space X is regular countably compact and if $X = \bigcup\{X_n : n < \omega\}$ with each X_n left-separated; moreover X will be compact then. Aiming for this result, Tkačenko

considered the situation in the whole generality, i.e. the space X was assumed to be τ-compact and $X = \cup\{X_\alpha : \alpha < \tau\}$ with each X_α left-separated (τ an infinite cardinal) and proved further results, some of which will be restated here.

The following notation will be frequently used throughout the whole paper: If $(A, <)$ is an ordered set and if $x \in A$, then $A(\leftarrow, x)$ denotes the initial segment $\{y \in A : y < x\}$. Similarly, $A(\leftarrow, x] = \{y \in A : y \le x\}$, $A(x, \rightarrow) = \{y \in A : y > x\}$, $A[x, \rightarrow) = \{y \in A : y \ge x\}$.

As usually adopted, cardinals are identified with the initial ordinals of the same cardinality.

1. __Definition__. Let X be a topological space, $(P, <)$ ordered subset of X, $F \subset X$. The set F is called to be __wide with respect to__ P if $F \cap \overline{P[x, \rightarrow)} \neq \emptyset$ for each $x \in P$.

2. __Lemma__. Let X be a topological space, let $(P, <_P)$ be a free sequence in X, $(M, <_M)$ left-separated subspace of X, F closed subset of X which is wide with respect to P. Assume moreover that for each point $x \in X$ there is some $p \in P$ with $x \in \overline{P(\leftarrow, p)}$.

Then there exists a closed set $F' \subset F$ which is wide wrt P and such that either $F' \cap M = \emptyset$ or F' is discrete and contained in M.

(Recall that $(P, <)$ is a free sequence in X if $<$ is a well-ordering of P such that $\overline{P(\leftarrow, x)} \cap \overline{P[x, \rightarrow)} = \emptyset$ whenever $x \in P$.)

__Proof__. By a transfinite induction we shall define the points $m_\alpha \in M$ and the points p_α, $q_\alpha \in P$ as follows:

$q_\alpha = \sup_P\{p_\beta : \beta < \alpha\}$, ($\sup_P \emptyset = <_P$ first element of P)

$m_\alpha = <_M$-first element of $M \cap F \cap \overline{P[q_\alpha, \rightarrow)}$,

$p_\alpha = <_P$-first element of P such that $m_\alpha \notin \overline{P[p_\alpha, \rightarrow)}$.

Let γ be the first ordinal such that the induction cannot continue.

Case 1. q_γ cannot be defined. That means, $\{p_\alpha : \alpha < \gamma\}$ is a cofinal sequence of $(P, <_P)$. Notice that the sequence $\{m_\alpha : \alpha < \gamma\}$ is free: Fix $\alpha < \gamma$, according to the choice of m_β's and q_β's we have $\{m_\beta : \beta < \alpha\} \subset \overline{P(\leftarrow, q_\alpha)}$ and $\{m_\alpha : : \alpha \leq \beta < \gamma\} \subset \overline{P[q_\alpha ; \rightarrow)}$. Since P is free, $\overline{P(\leftarrow, q_\alpha)} \cap \cap \overline{P[q_\alpha, \rightarrow)} = \emptyset$, thus $\overline{\{m_\beta : \beta < \alpha\}} \cap \overline{\{m_\beta : \alpha \leq \beta < \gamma\}} = \emptyset$.

Put $H = \overline{\{m_\alpha : \alpha < \gamma\}}$ and consider the set $H - \{m_\alpha : : \alpha < \gamma\}$. If $H - \{m_\alpha : \alpha < \gamma\}$ is not wide wrt P, there exists some $p \in P$ with $(H - \{m_\alpha : \alpha < \gamma\}) \cap \overline{P[p, \rightarrow)} = \emptyset$. Now it is self-evident that the set $F' = \{m_\alpha : \alpha < \gamma\} \cap \overline{P[p, \rightarrow)}$ is closed, discrete, wide with respect to P and contained in $F \cap M$.

If $H - \{m_\alpha : \alpha < \gamma\}$ is wide wrt P, define $F' = H - \{m_\alpha : : \alpha < \gamma\}$. We have to verify that $F' \cap M = \emptyset$. Pick arbitrary $m \in M$ and let $\beta_0 = \sup \{\beta : m_\beta <_M m\}$. If $m_{\beta_0} = m$, then $m \notin F'$ trivially. Further, $m \notin \overline{M(\leftarrow, m)}$ since M is left-separated, hence $m \notin \overline{\{m_\beta : \beta < \beta_0\}}$. Finally, $m \notin \overline{\{m_\beta : \beta_0 \leq \beta < \gamma\}}$: Suppose not. Then $m \in \overline{P[q_{\beta_0}, \rightarrow)} \cap F \cap M$, the possibility $m = m_{\beta_0}$ was discussed and if $m <_M m_{\beta_0}$, we obtain a contradiction to the choice of m_{β_0}.

Case 2. m_γ cannot be defined. That means $M \cap F \cap \overline{P[q_\gamma, \rightarrow)} = \emptyset$. It suffices to define $F' = F \cap \overline{P[q_\gamma, \rightarrow)}$. The verification that the set F' is as required may be left to the reader.

Case 3. p_γ cannot be defined. This case is empty because of the assumption that each point $x \in X$ belongs to some

$\overline{P(\leftarrow,p)}$ and by the fact that P is free.

3. Lemma. Let τ be an infinite cardinal, X τ-compact topological space, $P = \{p_\alpha : \alpha < \tau^+\}$ dense subset of X. Then the space $\tilde{X} = \{x \in X$: there is $\alpha < \tau^+$ such that $x \in \overline{\{p_\beta : \beta < \alpha\}}\}$ is τ-compact.

The easy proof is omitted.

4. Theorem (Tkačenko [T_1]). Let τ be an infinite cardinal, let X be a τ-compact topological space, $X = \bigcup\{M_\alpha : \alpha < \tau\}$ where each M_α is a left-separated subspace of X. Then there does not exist a free sequence of length τ^+ in X, in particular, $t(X) \le \tau$.

(Recall that $t(X)$, the tightness of X, is $\inf\{\varkappa$: $\varkappa$ is a cardinal and $\forall Y \subset X \ \ \forall x \in \overline{Y} \ \ \exists Z \subset Y \ (x \in \overline{Z} \ \& \ |Z| \le \varkappa)\}$.)

Proof. Suppose the contrary: let $P = \{p_\alpha : \alpha < \tau^+\}$ be the free sequence in X. Being closed in X, the set $\overline{P}$ is τ-compact. By the lemma 3, the space $Y = \{x \in \overline{P}$: there is $\alpha < \tau^+$ with $x \in \overline{\{p_\beta : \beta < \alpha\}}\}$ is τ-compact, too.

Let $K_\alpha = M_\alpha \cap Y$ for $\alpha < \tau$; K_α is clearly left-separated, and $Y = \bigcup\{K_\alpha : \alpha < \tau\}$. We shall successively apply Lemma 2: Let $F_0 = Y$. F_0 is wide wrt P, closed in Y, K_0 is left-separated subspace of Y, thus there is an $F_1 \subset F_0$ which is closed, wide wrt P and either $F_1 \cap K_0 = \emptyset$ or $F_1 \subset K_0$ and F_1 is discrete. Clearly each set in Y which is wide wrt P is of cardinality at least τ^+, this fact together with the τ-compactness of Y rules out the second possibility. Hence $F_1 \cap K_0 = \emptyset$.

Proceeding by an obvious induction, we obtain on each successor stage $\alpha + 1$ a closed set $F_{\alpha+1} \subset F_\alpha$ such that $F_{\alpha+1} \cap K_\alpha = \emptyset$ and $F_{\alpha+1}$ is wide with respect to P. If $\alpha < \tau$ is a limit ordinal, define $F_\alpha = \bigcap\{F_\beta : \beta < \alpha\}$. Assuming all F_β ($\beta < \alpha$) to be wide wrt P, F_α will be wide wrt P, too: If $p_\xi \in P$, then $\{F_\beta \cap \overline{P[p_\xi, \rightarrow)} : \beta < \alpha\}$ is a decreasing sequence of closed sets in Y and Y is τ-compact, thus $F_\alpha \cap \overline{P[p_\xi, \rightarrow)}$ is non-void.

We have constructed a nested sequence $\{F_\alpha : \alpha < \tau\}$ of non-empty closed subsets of Y. Its intersection is empty, since $Y = \bigcup\{K_\alpha : \alpha < \tau\}$ and $K_\alpha \cap F_{\alpha+1} = \emptyset$ for each $\alpha < \tau$. But the space Y is τ-compact - a contradiction.

5. Definition. Let X be a topological space. Define

$\mathfrak{z}(X) = \inf\{|\mathcal{M}| : X = \bigcup\mathcal{M}$ and each $M \in \mathcal{M}$ is a left-separated subspace of X$\}$

$n(X) = \inf\{|\mathcal{D}| : \mathcal{D}$ is a family of nowhere dense sets in X such that $\bigcup\mathcal{D}$ contains all non-isolated points of X$\}$

6. Theorem. Let X be a dense-in-itself topological space such that $d(X)\cdot t(X) < n(X)$. Then $\mathfrak{z}(X) \geq n(X)$.

Proof. Choose a cardinal τ with $d(X)\cdot t(X) \leq \tau < n(X)$. We want to show that $\tau < \mathfrak{z}(X)$. Suppose the contrary: Let $\mathcal{M}$ be a family of left-separated subspaces of X such that $|\mathcal{M}| \leq \tau$ and $\bigcup\mathcal{M} = X$. Since $n(X) > \tau$, there must be some $M \in \mathcal{M}$ which cannot be covered by $\leq \tau$ nowhere dense subsets of X. Define $N = M(\leftarrow, a)$, where $a = \inf_M\{b \in M : M(\leftarrow, b)$ cannot be covered by $\leq \tau$ nowhere dense subsets of X$\}$

if such an a can be found, if not, let

$N = M$.

Clearly, the set N is not nowhere dense; let $K = N \cap \operatorname{int} \overline{N}$. Denote by $<_K$ the well-ordering of K induced by the order of M.

The following are easy observations:

(a) K cannot be covered by $\leq \tau$ nowhere dense subsets of X

(Notice that N has this property and that $N - K = N - (N \cap \cap \operatorname{int} \overline{N}) \subset \overline{N} - \operatorname{int} \overline{N}$, which is nowhere dense in X.)

(b) K is dense in int $\overline{N}$ (any nonvoid open set $U \subset \operatorname{int} \overline{N}$ meets N, hence $\emptyset \neq U \cap N = U \cap \operatorname{int} \overline{N} \cap N = U \cap K$).

Claim: The cofinality of $(K, <_K)$ is not greater than τ.

To prove the claim, choose some set $\{q_\xi : \xi < \tau\} \subset \operatorname{int} \overline{N}$ dense in int $\overline{N}$. Since $d(X) \leq \tau$, it is possible.

Since K is dense in int $\overline{N}$ and since $t(X) \leq \tau$, choose for each $\xi < \tau$ a set $T_\xi \subset K$ such that $|T_\xi| \leq \tau$ and $q_\xi \in \overline{T}_\xi$. Denote by T the union $\bigcup \{T_\xi : \xi < \tau\}$. Then $|T| \leq \tau$ and $\overline{T} \supset \overline{\{q_\xi : \xi < \tau\}} \supset K$. It follows that T is cofinal in K: If not, for $t = \sup_K T$ we have that $t \in \overline{T} \subset \overline{K(\leftarrow, t)}$, but K is left-separated - a contradiction.

Having proved the claim, let us choose a cofinal subset $\{m_\xi : \xi < \tau\}$ of K. We obtain $K \subset \bigcup \{K(\leftarrow, m_\xi) : \xi < \tau\} \subset \subset \bigcup \{N(\leftarrow, m_\xi) : \xi < \tau\}$. By the choice of N, for each $\xi < \tau$ there is a family $\mathcal{A}_\xi$ of nowhere dense subsets of X, such that $|\mathcal{A}_\xi| \leq \tau$ and $\bigcup \mathcal{A}_\xi \supset N(\leftarrow, m_\xi)$. Then $K \subset \bigcup \{\bigcup \mathcal{A}_\xi : : \xi < \tau\}$, which contradicts (a).

7. <u>Corollary</u> (Tkačenko [T_2]): Let X be a compact Hausdorff space, $X = \bigcup \{M_n : n < \omega\}$, where each M_n is a left-separated subspace of X. Then X is scattered.

<u>Proof</u>. It suffices to show that X has at least one isolated point. Suppose the contrary: let X be dense-in-itself. Then X can be continuously mapped onto 2^ω; let f be such a mapping. Choose $Y \subset X$ to be a closed subspace of X such that $f \restriction Y$ is irreducible. Then Y is a compact Hausdorff space

without isolated points which admits a continuous irreducible mapping onto 2^{ω}. This implies $d(Y) = d(2^{\omega}) = \omega$, $n(Y) = n(2^{\omega}) > \omega$. Moreover, $\xi(X) = \omega$ and X is (countably) compact, according to Theorem 4, $t(X) \leq \omega$, hence $t(Y) \leq \omega$. Applying Theorem 6, we obtain $\xi(Y) \geq n(Y) > \omega$. But $\omega \geq \geq \xi(X) \geq \xi(Y)$ - a contradiction.

8. <u>Concluding remarks</u>. (a) There exists an example of a (compact Hausdorff) topological space X without isolated points, where $\xi(X) \cdot t(X) \cdot d(X) < |X|$ holds. Thus the number $n(X)$ cannot be replaced by $|X|$ in Theorem 6.

(b) The original Tkačenko's proofs heavily depend on the fact that the following statement is true for some particular choices of the spaces X and Y: If X and Y are (regular) topological spaces and $f: X \longrightarrow Y$ a continuous perfect irreducible onto mapping, then $\xi(X) \geq \xi(Y)$. It suggests a question: Is the statement true in general?

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