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A LIMIT THEOREM FOR FUNCTIONALS OF A POISSON PROCESS

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Abstract: Let μ be a random point measure defined on a locally compact topological space X with countable basis and let μ have the Poisson distribution Q_ν with intensity measure ν. The asymptotic behaviour of the distribution function of the random variable $Z_n(\mu)=\mu(hI_{K_n})$ as the compact subset $K_n \uparrow X$ is considered. This work also deals with the rate of convergence to the limit distribution.

Key words: Stochastic point process, asymptotic normality, intensity measure, exponential trend.

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§ 1. Introduction. Poisson processes form an important class of point processes. Many interesting problems of statistical analysis of Poisson processes on the line have been considered in [1] by D.R. Cox and P.A.W. Lewis and (on more general spaces) by M. Brown [3]. This article is concerned with the limit distribution of certain linear functionals of a Poisson process. Limit theorems will be stated in Section 2. The rate of convergence to the limit distribution function will be considered in Section 3. Section 4 contains some applications of the results obtained in Section 2.

§ 2. **Limit theorem.** Following [4],[5] let us consider a locally compact topological space X with countable basis. Let $\mathcal{B}(X)$ be the σ-algebra of Borel subsets of X, $\mathcal{M} = \mathcal{M}(X)$ the family of Radon measures on $(X,\mathcal{B}(X))$ and $\mathcal{K}_c$ - the class of continuous functions with compact supports defined on X.

Let us also consider a Poisson process Q_ν on X with intensity measure ν ($\nu \in \mathcal{M}(X)$), i.e., a probability distribution defined on the σ-algebra $\mathcal{C}(\mathcal{M})$ generated by all open subsets with respect to the topology of vague convergence x) with the characteristic functional defined by

$$(1) \quad \hat{Q}_\nu(f) = \int_{\mathcal{M}} \exp(i\mu(f))\, Q_\nu(d\mu) = \exp(\nu(e^{if}-1)), \quad f \in \mathcal{K}_c,$$

where $\nu(f) = \int_X f(x)\,\nu(dx)$.

Suppose that $\mu \in \mathcal{M}$ is a realization of Q_ν. Usually one can only observe the realization μ on some compact set K of X, as X too large.

Let us consider a statistic of the form

$$(2) \qquad Z_K(\mu) = \mu(hI_K),$$

where I_K is the indicator of K, h is some measurable function on X.

The statistic $Z_K(\mu)$ plays an important role for many problems of testing hypothesis and estimating the parameters of Poisson processes. The distribution function of $Z_K(\mu)$ depends on h, K and ν, and is rather complicated, the asympto-

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x) $\{\mu_n\}$ is called to be vaguely convergent to μ iff $\mu_n(f) \longrightarrow \mu(f)$ for all $f \in \mathcal{K}_c$.

tic theory for such statistics is therefore convenient for practical purposes.

Suppose that K_n is a sequence of compact sets such that $K_n \uparrow X$. Let $Z_n(\mu) = Z_{K_n}(\mu)$, and let us consider the asymptotic behaviour of the distribution law under Q_ν of the random variable of the form

(3) $$Y_n(\mu) = (Z_n(\mu) - a_n)/b_n,$$

where a_n, b_n ($b_n > 0$, for all n) are constants.

Note that

$$Z_n(\mu) = \pm\infty \quad \text{iff } A = \{x : h(x) = \pm\infty\} \subset K_n \cap \operatorname{supp} \mu .$$

Consequently, letting

$$R_n = \{\mu : \mu(hI_{K_n}) = Z_n(\mu) \neq \pm\infty\}$$

we obtain (see [4])

$$Q_\nu(R_n) = \exp(-\nu(K_n A)).$$

Consequently, for the existence of the limit distribution of $Y_n(\mu)$ the necessary condition is

$$Q_\nu\{Z_n(\mu) = \pm\infty\} = 1 - Q_\nu(R_n) = 1 - \exp(-\nu(AK_n)) \longrightarrow 1 - \exp(-\nu(A)) = 0$$

or

(4) $$\nu(AK_n) \longrightarrow \nu(A) = \nu\{x : h(x) = \pm\infty\} = 0$$

Therefore, in the following theorems we always assume that (4) is fulfilled.

Let $\lambda_n = \nu(K_n)$, $\nu_K(\cdot)$ be the restricted measure of ν on K, i.e. $\nu_K(A) = \nu(AK)$, for all K and $A \in \mathcal{B}(X)$, and $\nu_n(\cdot) = \nu_{K_n}(\cdot)$, $G_n(t)$ be the characteristic function ch.f. of $Z_n(\mu)$ under Q_ν .

We have the following theorems

Theorem 1. Assume that $\lambda = \nu(X) < \infty$, then

(5) $$G_n(t) \longrightarrow \exp(\lambda[g(t) - 1]) = G(t), \text{ say,}$$

holds, where

(6) $$g(t) = \nu(\exp(ith))/\lambda$$

is the ch.f. of random variable $h(T)$ with T being a random element in X possessing the distribution law $\nu(\cdot)/\lambda$.

Further, $G(t)$ is the ch.f. of the random variable $\gamma\xi$, where γ is some constant, ξ has the Poisson distribution with the mean value λ, iff $h = \gamma$, ν-a.e. . $G(t)$ is always the ch.f. of a nonnormal random variable.

The case $\lambda = \infty$ is more interesting.

Theorem 2. Suppose that $\lambda = \infty$. Then the following conditions (i),(ii) are sufficient for the existence of number sequences $\{a_n\}$ and $\{b_n\}$ with $b_n \longrightarrow \infty$ such that

(7) $$F_n(y) = Q_\nu\{Y_n(\mu) < y\} = Q_\nu\{(Z_n(\mu) - a_n)/b_n < y\} \longrightarrow F(y)$$

where, here and in the sequel, the convergence is meant in the weak sense.

(i) $a_n/b_n \lambda_n^{1/2} \longrightarrow \alpha$ (α - finite)

(ii) $P\{(S_n - a_n)/b_n < y\} \longrightarrow K(y)$

where

$$S_n = \sum_{k=1}^{[\lambda_n]} h(x_{nk})$$

is the sum of independent random variables $h(x_{nk})$ with x_{nk}, $k=1,2,\ldots,[\lambda_n]$, being identically distributed independent random elements in X possessing the common distribution law $\nu_n(\cdot)/\lambda_n$ for each n and with $[\lambda_n]$ denoting the entire of λ_n.

Further, F has the form

(8) $$F(y) = K * \Phi_{\alpha}(y)$$

where $\Phi_{\alpha}(y)$ is the normal distribution function with mean value zero and variance α^2.

Notice. $\Phi_0(y)$ is the distribution function with jump one at zero, whereas $\Phi_1(y)$ is redenoted by $\Phi(y)$.

Proof of Theorem 1. It is easy to see that the ch.f. $G_n(t)$ of Z_n under Q_ν is defined by

(9) $$\begin{aligned} G_n(t) &= E_{Q_\nu}(\exp(it\mu(hI_{K_n}))) = \hat{Q}_{\nu_n}(th) \\ &= \exp(\nu_n(\exp(ith) - 1)) = \exp(\lambda_n[g_n(t) - 1]) \end{aligned}$$

with

(10) $$g_n(t) = \nu_n(\exp(ith))/\lambda_n$$

Since $\lambda_n \longrightarrow \lambda$ as $K_n \uparrow X$, $g_n(t)$ converges to $g(t)$ and (5) follows from (9).

The second statement of Theorem 1 comes true iff $g(t) = \exp(i\gamma t)$, but this occurs iff $h(x) = \gamma$, ν-a.e..

As to the last statement, let us suppose inversely that $G(t) = \exp(iat - b^2t^2/2)$, then $g(t) = 1 + iat/\lambda - b^2t^2/2\lambda$.

However, the right hand side of this equality is not a ch.f.. This proves the last statement.

Proof of Theorem 2. Let V_n be a Poisson distributed random variable with mean λ_n. For the sake of simplicity we suppose that λ_n is an integer.

Put $\bar{A}_n(y) = P\{V_n < y\}$

$$A_n(y) = P\{(V_n - \lambda_n)/\lambda_n^{1/2} < y\} = \bar{A}_n(y\lambda_n^{1/2} + \lambda_n)$$

It is obvious that $A_n(y) \longrightarrow \Phi(y)$ since $\lambda_n \rightarrow \lambda = \infty$.

It follows from (9) that

(11) $$G_n(t)=\exp(-\lambda_n)\sum_{k=0}^{\infty}\frac{\lambda_n^k}{k!}[g_n(t)]^k = \int_0^{\infty}[g_n(t)]^y d\,\bar{A}_n(y)=\int_{-\infty}^{\infty}[g_n(t)]^{y\lambda_n^{1/2}+\lambda_n}\,dA_n(y)$$

Let k(t), f(t) be the ch.f. corresponding to K, F and $H_n(t)$, $k_n(t)$ be the ch.f. of Y_n, $(S_n-a_n)/b_n$, respectively. It is easy to see from (11) that

$$H_n(t)=\exp(-ita_n/b_n)\,G_n(t/b_n) = \int_{-\infty}^{\infty}[k_n(t)]^{1+y/\lambda_n^{1/2}}\exp(itya_n/b_n\lambda_n^{1/2})\,dA_n(y)\longrightarrow$$

$$\longrightarrow k(t)\int_{-\infty}^{\infty}\exp(i\alpha ty)\,d\phi(y)=k(t)\exp(-\alpha^2t^2/2)$$

This proves Theorem 2.

__Remark__. According to Theorem 2 the problem of investigating the convergence of $F_n(y)$ reduces to the classical limit problem for the sum S_n of independent random variables, and with the aid of this theorem we can obtain a large class of limit distributions of Z_n.

The following theorem states conditions for asymptotic normality of Z_n.

We say that Z_n is asymptotically normal $N(a_n,b_n^2)$ if

$$\sup_{-\infty<y<\infty}|F((Z_n-a_n)/b_n< y) - \phi(y)|\longrightarrow 0$$

__Theorem 3__. Assume that $\lambda_n\longrightarrow\infty$. Then necessary and sufficient conditions for the existence of number sequences $\{a_n\}$ and $\{b_n\}$ with $b_n>0$ and $b_n\longrightarrow\infty$ such that

(i) $g_n(\varepsilon)=\nu(K_n\cap\{x:|h(x)|>\varepsilon b_n\})\longrightarrow 0$ as $n\longrightarrow\infty$ for all $\varepsilon>0$

(ii) Z_n is asymptotically normal $N(a_n,b_n^2)$

are that there exists a number sequence $\{d_n\}$ with $d_n \longrightarrow \infty$ such that

(a) $C_n^2 = \nu(h^2 I_{K_n S_n}) \longrightarrow \infty$ where $S_n = \{x : |h(x)| \le d_n\}$

(b) $d_n = o(C_n)$, $\nu(K_n S_n^c) \longrightarrow 0$

Further, in this case the constants a_n, b_n can be defined by

$$a_n = \nu(hI_{K_n S_n}),\quad b_n^2 = C_n^2 = \nu(h^2 I_{K_n S_n}). \tag{12}$$

<u>Proof of necessity</u>. Suppose that (i),(ii) are fulfilled. Since $g_n(\varepsilon) \longrightarrow 0$, there exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$ and $g_n(\varepsilon_n) \longrightarrow 0$.

Putting $d_n = \varepsilon_n b_n = o(b_n)$, we obtain $\nu(K_n S_n^n) \longrightarrow 0$

Further, the logarithm of the ch.f. $H_n(t)$ of $Y_n = (Z_n - a_n)/b_n$ can be extended in the following form (see (9))

$$\begin{aligned} \ln H_n(t) &= -ita_n/b_n + \nu([\exp(ith/b_n) - 1]I_{K_n}) = \\ &-ita_n/b_n + \nu([\exp(ith/b_n) - 1]I_{K_n S_n}) + o(1) \end{aligned} \tag{13}$$

since

$$|\nu([\exp(ith/b_n) - 1]I_{K_n S_n^c})| \le 2\nu(K_n S_n^c) \longrightarrow 0$$

Furthermore,

$$\begin{aligned} \nu([\exp(ith/b_n)-1]I_{K_n S_n}) &= it\,\nu(hI_{K_n S_n})/b_n - \\ &- t^2 \nu(h^2 I_{K_n S_n})/2b_n^2 + \theta |t|^3 (d_n/b_n)\, \nu(h^2 I_{K_n S_n})/6b_n^2 \end{aligned} \tag{14}$$

with $|\theta| \le 1$.

It follows from (13),(14) and from the assumption of asymptotic normality of Z_n that

$$\ln H_n(t) = -ita_n/b_n + it\,\nu(hI_{K_nS_n})/b_n - t^2\nu(h^2I_{K_nS_n})/2b_n^2 + \tag{15}$$
$$\theta|t|^3(d_n/6b_n^3)\,\nu(h^2I_{K_nS_n}) + o(1) \longrightarrow -t^2/2.$$

(15) holds iff

$$\nu(h^2I_{K_nS_n})/b_n^2 \longrightarrow 1, \text{ or } C_n^2/b_n^2 \longrightarrow 1$$

and it follows from $d_n/b_n \longrightarrow 0$ that $d_n/C_n \longrightarrow 0$.

Proof of sufficiency. Suppose that (a),(b) are satisfied. Then putting in (15) $a_n = \nu(hI_{K_nS_n})$, $b_n = C_n$, we obtain

$$\ln H_n(t) \longrightarrow -t^2/2$$

i.e. (ii) is fulfilled, whereas (i) follows immediately from (b) with $b_n = C_n$.

Remark. The statement on the sufficiency of conditions (a),(b) of Theorem 3 may be considered as a corollary of Theorem 2.

Indeed, according to Theorem 2.3 in [2], $F_n(y) = P\{(Z_n - a_n)/b_n < y\} \longrightarrow \phi(y)$ iff for any subsequence $\{n'\}$ of $\{n\}$ there exists a subsequence $\{k\}$ of $\{n'\}$ such that $F_k(y) \longrightarrow \phi(y)$. We shall show that the statement holds, provided (a),(b) are satisfied.

Note that if a_n, b_n are given by (12) we have

$$|a_n/b_n\,\lambda_n^{1/2}| \le 1$$

The logarithm of the ch.f. $k_n(t)$ of $(S_n - a_n)/b_n$ defined in Theorem 2 is given by

$$\ln k_n(t) = -ita_n/b_n + \lambda_n \ln g_n(t), \tag{16}$$

where

$$g_n(t) = \nu(\exp(ith/b_n)I_{K_n})/\lambda_n. \tag{17}$$

On the other hand,

$$|\nu(\exp(ith/b_n)I_{K_nS_n^c})| \leq \nu(K_nS_n^c) \longrightarrow 0$$

hence

$$g_n(t) = \nu(\exp(ith/b_n)I_{K_nS_n})/\lambda_n + o(\lambda_n^{-1}) =$$

(18) $$= 1 + ita_n/b_n\lambda_n - t^2/2\lambda_n + O(d_n/b_n\lambda_n) + o(\lambda_n^{-1}) =$$

$$= 1 + ita_n/b_n\lambda_n - t^2/2\lambda_n + o(\lambda_n^{-1}).$$

From (16) - (18) we obtain easily

(19) $$\ln k_n(t) = -t^2/2 + t^2a_n^2/2\lambda_n b_n^2 + o(1).$$

On the other hand, for any subsequence $\{n'\}$ of $\{n\}$ there exists a subsequence $\{k\}$ of $\{n'\}$ such that $a_k^2/\lambda_k b_k^2 \longrightarrow \alpha^2$, hence

(20) $$\lim_{k\to\infty} \ln k_k(t) = (\alpha^2-1)t^2/2, \quad \alpha^2 \leq 1.$$

Consequently, by Theorem 2, $F_k(y) \longrightarrow K * \Phi_\alpha(y)$, where K is the distribution function corresponding to the ch.f., the logarithm of which is equal to the right hand part of (20). The logarithm of the ch.f. of $K * \Phi_\alpha(y)$ is therefore equal to

$$(\alpha^2-1)t^2/2 - \alpha^2t^2/2 = -t^2/2.$$

Consequently, $K * \Phi_\alpha(y) = \Phi(y)$. This proves the "sufficiency" part of Theorem 3.

<u>Corollary 1</u>. Assume that

$$b_n^2 = \nu(h^2 I_{K_n}) < \infty, \quad b_n \longrightarrow \infty \quad \text{and} \quad \sup_{x\in K_n} |h(x)| = o(b_n).$$

Then Z_n is asymptotically normal $N(a_n, b_n^2)$ with $a_n = \nu(hI_{K_n})$.

<u>Proof</u>. Corollary 1 follows immediately from Theorem 3

by putting

$$d_n = \begin{cases} \sup_{K_n} |h(x)| & \text{if} \quad \sup_{K_n} |h(x)| \longrightarrow \infty \\ b_n^{1/2} & \text{if} \quad \sup_{K_n} |h(x)| \not\longrightarrow \infty \end{cases}$$

<u>Corollary 2</u>. (Theorem of Brown (1972).) Let $\nu_1, \nu_2, \wp$ be Radon measures on $(X, \mathcal{B}(X))$ and $\nu_1, \nu_2 \ll \wp$. Further, suppose that the following conditions (i),(ii),(iii), or (i),(ii),(iv) are satisfied:

(i) $\nu_2 \ll \nu_1$, $f_1 = d\nu_1/d\wp$, $f_2 = d\nu_2/d\wp$.

(ii) There exists a finite positive number M such that $\nu_1\{x: |\ln(f_2/f_1(x))| > M\} < \infty$.

(iii) $\nu_1([(f_2/f_1)^2 - 1]^2 I_{D_c}) = \infty$ for all $c > 0$,

where

$$D_c = \{x: |[f_2(x)/f_1(x)]^2 - 1| < c\}.$$

(iv) There exists a finite number M_0 such that $\nu_1\{x: |\ln(f_2/f_1)| \geq M_0\} = \infty$.

Then, as $K_n \uparrow X$, $\mu(I_{K_n} \ln(f_2/f_1))$ is asymptotically normal $N(a_n, b_n^2)$ under Q_{ν_1}, where

$$a_n = \nu_1(\ln(f_2/f_1) I_{K_n S_M}), \quad b_n^2 = \nu_1(\ln^2(f_2/f_1)\, I_{K_n S_M})$$

with $S_M = \{x: |\ln(f_2/f_1)| \leq M\}$.

<u>Proof</u>. Corollary 2 can be obtained immediately from Theorem 3 by putting $h = \ln(f_2/f_1)$.

Indeed, for $\nu_2 \ll \nu_1$, $\ln(f_2/f_1)$ is well defined ν_1-a.e.. Let us now suppose that (ii),(iii) hold, then

$(f_2/f_1)^2 - 1 \sim 2\ln(f_2/f_1)$ as $|(f_2/f_1)^2 - 1| < c$, hence it follows from (iii) that $\nu_1(h^2 I_{S_{c/2}}) = \infty$ and

(21) $$\nu_1(h^2 I_{K_n S_M}) \geq \nu_1(h^2 I_{K_n S_c}) \longrightarrow \infty \quad \text{if } c \leq M$$

If (ii),(iv) hold, then $M_o < M$ and

$$b_n^2 = \nu_1(h^2 I_{K_n S_M}) \geq \int_{K_n \cap \{M_o \leq |h| \leq M\}} h^2 \nu_1(dx) \geq$$

(22) $$\geq M_o^2 \nu_1(\{M_o \leq |h| \leq M\} \cap K_n) = M_o^2 [\nu_1(\{|h| \geq M_o\} \cap K_n) -$$

$$- \nu_1(\{|h| \geq M\} \cap K_n)] \longrightarrow \infty$$

Consequently, choosing $d_n = o(b_n)$, $d_n \longrightarrow \infty$, then it follows from (21),(22) that

$$\nu_1(h^2 I_{K_n S_{d_n}}) \longrightarrow \infty .$$

Further, $\nu_1(K_n S^c_{d_n}) \leq \nu_1(S^c_{d_n}) \longrightarrow 0$, since $\nu_1(S^c_M) = \sum_{i=1}^{\infty} \nu_1(d_i < |h| \leq d_{i+1}) < \infty$ implies $\nu_1(S^c_{d_n}) = = \sum_{j=n}^{\infty} \nu_1(d_j < |h| \leq d_{j+1}) \longrightarrow 0$, letting $M = d_1 < d_2 < \dots$. Thus the conditions (a),(b) of Theorem 3 are satisfied. The condition ν_1 $(h = \pm\infty) = 0$ is also fulfilled since

$$\nu_1 \ (h = \pm\infty) \leq \nu_1(S^c_{d_n}) \longrightarrow 0.$$

Consequently, the statements of Corollary follows from Theorem 3.

<u>Remark 1</u>. We observe that the assumptions of Theorem 3 are strictly weaker than those of the cited theorem of Brown.

In fact, let ν_1 be Lebesgue measure on the half line $X = [0,\infty)$, $\nu_2 \ll \nu_1$ with $d\nu_2/d\nu_1 = \exp(t) = f_2(t)$, $f_1(t) \equiv 1$.

Then $h(t)=\ln f_2(t)=t$. It is obvious that condition (ii) of the theorem of Brown is not fulfilled, since

$$\nu_1\{|h|>M\}=\nu_1\{t:t>M\}=\infty \quad \text{for all } M>0.$$

Theorem 3 is, however, utilizable. Indeed, if $K_n=[0,T_n]$ with $T_n\uparrow\infty$, letting $T_n=d_n$ we have $K_n\cap S_n^c=\emptyset$,

$$b_n^2=\int_0^{d_n} t^2dt = d_n^3/3, \text{ so that } d_n = o(b_n).$$

Consequently, by Theorem 3, Z_n is asymptotically normal $N(a_n,b_n^2)$ with

$$a_n=\int_0^{T_n} t\,dt = T_n^2/2,\ b_n^2 = T_n^3/3.$$

§ 3. The rate of convergence to limit distribution.

Theorem 4. Suppose that $\nu(|h|^3 I_{K_n})<\infty$ and let

$$a_n=\nu(hI_{K_n}),\ b_n^2=\nu(h^2I_{K_n}),\ \gamma_n=\nu(|h|^3 I_{K_n}),$$

$$b_n'^2=b_n^2/\lambda_n;\ \gamma_n'=\gamma_n/\lambda_n,\ F_n(y)=Q_\nu\{(Z_n-a_n)/b_n<y\}.$$

Then

$$\sup_{-\infty<y<\infty}|F_n(y)-\phi(y)|\leq A\,\gamma_n/b_n^3=A\,\gamma_n'/b_n'^3\,\lambda_n^{1/2} \tag{23}$$

where A may be taken the value

$$A = (3/2\pi)^{1/2}+C^2(1/\pi)/(2\pi)^{3/2}$$

with C(t) being the solution of the equation

$$\int_0^{C(t)}(\sin^2u/u^2)du = \pi/4 + 1/8t$$

Proof. Let $H_n(t)$ be the ch.f. corresponding to F_n. Then (see (9))

$$H_n(t)=\exp\{-ita_n/b_n+\nu([\exp(ith/b_n)-1]\, I_{K_n})\}=$$

$$=\exp\{-ita_n/b_n+it\,\nu(hI_{K_n})-t^2\nu(h^2I_{K_n})/2b_n^2+\theta|t|^3\gamma_n/6b_n^3\}=$$

$$=\exp(-t^2/2+\theta|t|^3\gamma_n/6b_n^3) \text{ with } |\theta|\leq 1,$$

hence

$$|H_n(t)-\exp(-t^2/2)|=\exp(-t^2/2)\,|\exp(\theta|t|^3\gamma_n/6b_n^3)-1|\leq$$

$$\leq(|t|^3\gamma_n/6b_n^3)\exp(-t^2/2+|t|^3\gamma_n/6b_n^3)\leq|t|^3\exp(-t^2/6)\,\gamma_n/6b_n^3$$

provided $|t|\leq 2b_n^3/\gamma_n=T$, say.

In accordance with Theorem 2, p. 137,[6], we have

$$\sup_y|F_n(y)-\phi(y)|\leq$$

$$\leq\frac{1}{\pi}\int_{-T}^{T}\left|\frac{H_n(t)-\exp(-t^2/2)}{t}\right|dt+C^2(1/\pi)/T\pi\sqrt{2\pi}\ .$$

We therefore receive from (25)

$$\sup_y|F_n(y)-\phi(y)|\leq(\gamma_n/6\pi b_n^3)\int_{-\infty}^{\infty}t^2\exp(-t^2/6)dt+$$

$$+C^2(1/\pi)\,\gamma_n/b_n^3(2\pi)^{3/2}=A\,\gamma_n/b_n^3.$$

Remark 2. If $\gamma'_n/b_n'^3\leq M$ (in general, it is usually fulfilled), then

$$\sup_y|F_n(y)-\phi(y)|\leq AM\,\lambda_n^{-1/2}$$

and this is the best estimation of the deviation between $F_n(y)$ and $\phi(y)$. Indeed, if $h\equiv 1$ then $Z_n(\mu)=\mu(K_n)$ has Poisson distribution with the mean value $\lambda_n=\nu(K_n)$ and it is to see that

$$\sup_y|F_n(y)-\phi(y)|=O(\lambda_n^{-1/2}) \text{ where } F_n(y)=Q_\nu\{[\mu(K_n)-\lambda_n]\,\lambda_n^{-1/2}<y\}$$

Example. Let us consider the example described in Remark 1. We have

$\lambda_n = T_n$, $\gamma_n' = T_n^3/4$, $b_n'^2 = T_n^2/3$, hence $\gamma_n'/b_n'^3 = 3\sqrt{3}/4$, thus, by (23), $\sup|F_n(y) - \phi(y)| \leq 3A\sqrt{3}/4T_n^{1/2}$.

§ 4. Some applications

1. Estimating the parameter of exponential trend. Let us consider a family of Poisson processes $\{Q_\theta = Q_{\wp_\theta},\ \theta \in \Theta\}$ on $(X, \mathcal{B}(X))$, where the intensity measure $\wp_\theta$ possesses the density with respect to some Radon measure λ

$d\wp_\theta/d\lambda = \exp(\theta T(x))$, $\theta \in \Theta$ - an open interval of R^1.

Usually we can only observe a realization μ of the process Q_θ on a compact set K_n of X. In this case let us consider the -algebra $\mathcal{A}_{K_n}$ generated by $\{\mu(A): A \subseteq K_n\}$. Then, according to [4] the restrictions $Q_\theta^{(n)}$, $Q_\lambda^{(n)}$ of Q_θ, Q_λ on $\mathcal{A}_{K_n}$ have the property that $Q_\theta^{(n)} \ll Q_\lambda^{(n)}$, and the logarithm of the likelihood function of the process is given by

(26) $$L_n(\theta) = \ln(dQ_\theta^{(n)}/dQ_\lambda^{(n)}) = \lambda(K_n) - \wp_\theta(K_n) + \theta\mu(TI_{K_n}).$$

Let

$$h_n(\theta) = \wp_\theta(K_n) = \lambda(I_{K_n}\exp(\theta T))$$

Suppose that $h_n(\theta)$ satisfied the following conditions:

(i) $dh_n(\theta)/d\theta = \lambda(I_{K_n} T \exp(\theta T)) = a_n(\theta)$, say, and $a_n(\theta)$ is finite,

(ii) $d^2h_n(\theta)/d\theta^2 = \lambda(I_{K_n} T^2 \exp(\theta T)) = b_n^2(\theta) < \infty$, and $b_n(\theta) \to \infty$ as $n \to \infty$,

(iii) $C_n(\theta) = \lambda(|T|^3 \exp(\theta T)\, I_{K_n})$ is finite and there exists a number $\delta(\theta) > 0$ such that

$$\sup \{ |C_n(\theta')|,\ |\theta' - \theta| < \delta \} / b_n^3(\theta) \to 0 \text{ as } n \to \infty$$

It is obvious that $\{d\, Q_\theta^{(n)}/dQ_\lambda^{(n)},\ \theta \in \Theta\}$ is an exponential family of one parameter and $Z_n(\mu) = \mu(TI_{K_n})$ is a complete sufficient statistic for θ and is an unbiased estimate of $a_n(\theta)$. In particular, $Z_n(\mu)$ takes in the form of the statistic considered in Theorem 2 and 3. We have the following statement:

Proposition. Assume that the above conditions (i), (ii), (iii) are satisfied. Then the likelihood equation $dI_n(\theta)/d\theta = 0$ or $a_n(\theta) - Z_n(\mu) = 0$ has under Q_{θ_0} unique solution $\hat\theta(\mu)$ as $n \to \infty$ and with probability approaching to 1, and $\hat\theta(\mu)$ is asymptotically normal $N(\theta_0, b_n^{-2}(\theta_0))$.

Proof. At first let us remark that according to (23) of Theorem 4

(27) $$\sup_y | Q_{\theta_0} \{ (Z_n - a_n)/b_n(\theta_0) < y \} - \phi(y) | \le A C_n(\theta_0)/b_n^3(\theta_0) \to 0$$

Further,

(28) $$a_n(\theta_0 \pm \delta) = a_n(\theta_0) \pm \delta b_n^2(\theta_0) + \beta \delta^2 C_n(\theta_0 + \alpha\delta)/2,$$
$$|\beta|, |\alpha| \le 1.$$

Choosing $\delta = u_n/b_n(\theta_0)$ so that $u_n/b_n \to 0$ and $u_n(\theta_0) \to \infty$, $u_n^2(\theta_0) = O(b_n^3/C_n)$ (this is always fulfilled) we obtain from (28)

$$\frac{a_n(\theta_0 \pm \delta) - Z_n(\mu)}{b_n(\theta_0)} = \frac{a_n(\theta_0) - Z_n(\mu)}{b_n(\theta_0)} \pm u_n(\theta_0) + O(1)$$

Consequently, the function $a_n(\theta) - Z_n$ will change its sign on the interval $(\theta_0-\delta,\theta_0+\delta)$. Furthermore, by (ii), for n sufficiently large $a_n(\theta)$ is strictly increasing, hence the likelihood equation has only solution $\hat{\theta}$. Further,

$$(29)\quad b_n(\theta_0)[\hat{\theta}-\theta_0]<t \iff a_n(\hat{\theta})<a_n(\theta_0+tb_n^{-1}) \iff$$
$$\iff Z_n(\mu)<a_n(\theta_0+tb_n^{-1}),$$

whereas $a_n(\theta_0+tb_n^{-1})$ can be extended in the form (see (28))

$$(30)\quad a_n(\theta_0+tb_n^{-1})=a_n(\theta_0)+tb_n(\theta_0)+\beta t^2 c_n(\theta_n+\alpha tb_n^{-1})/2b_n^2$$

It follows from (29),(30),(27) and (iii) that

$$Q_{\theta_0}\{b_n(\theta_0)[\hat{\theta}-\theta_0]<t\}=Q_{\theta_0}\{[Z_n(\mu)-a_n]/b_n<t\pm\beta t^2 c_n(\theta_0+$$
$$+\alpha tb_n^{-1})/2b_n^3\}\longrightarrow\phi(t)$$

as $n\to\infty$ for any t fixed. This proves the asymptotic normality of $Z_n(\mu)$.

<u>Example</u>. Let $X=[0,\infty)$, $K=[0,T_n]$ with $T_n\uparrow\infty$, $T(x)=x$, λ be Lebesgue measure, $\Theta=(0,\infty)$. Then $\hat{\theta}$ is the unique solution of the equation

$$\int_0^{T_n} x\exp(\theta x)dx=\int_0^{T_n} x\,\mu(dx)=Z_n(\mu),\ \text{say, or equivalently}$$

$$T_n\exp(\theta T_n)/\theta-[\exp(\theta T_n)-1]/\theta^2=Z_n(\mu)$$

and it is easy to verify that

$c_n(\theta')/b_n^3(\theta)\sim\theta^{3/2}\exp([\theta'-\theta-\theta/2]\,T_n)/\theta'\longrightarrow 0$ for all $\theta':|\theta'-\theta|<\theta/2=\delta(\theta)$. Consequently, by the above proposition $\hat{\theta}$ is asymptotically normal $N(\theta,b_n^{-2}(\theta))$ under Q_θ with $b_n^2(\theta)\approx\approx T_n^2\exp(\theta T_n)/\theta$.

<u>Remark</u>. By the theorem of Rao - Blackwell and by the

above proposition estimate $\hat{\theta}$ of θ is asymptotically efficient.

2. Distinguishing two Poisson processes. Let us consider two Poisson processes Q_{ν_1}, Q_{ν_2} and assume that $\nu_1, \nu_2 \ll \ll \lambda$. Further, suppose that we have a realization of μ only on compact subset K at our disposal. Let $\mathcal{A}_K$ be σ-algebra generated by $\{\mu(A): A\subset K\}$. Then (see [4]) the restrictions $Q_{\nu_{iK}}$, Q_{λ_K} of Q_{ν_i}, Q_λ on $\mathcal{A}_K$, i=1,2, respectively, have the property that $Q_{\nu_{iK}} \ll Q_{\lambda_K}$ and

$$dQ_{\nu_{iK}}/dQ_{\lambda_K} = \exp\{\lambda(K) - \nu_i(K) + \mu(I_K \ln(d\nu_i/d\lambda))\},\ i=1,2.$$

Consequently, for testing Q_{ν_1} against Q_{ν_2} we can employ the likelihood ratio test, under which Q_{ν_1} will be rejected if

$$\frac{\exp[\lambda(K) - \nu_2(K) + \mu(\ln(d\nu_2/d\lambda)I_K)]}{\exp[\lambda(K) - \nu_1(K) + \mu(\ln(d\nu_1/d\lambda)I_K)]} > C$$

or equivalently

$$\mu(hI_K) > C_\alpha$$

where $h=\ln\left(\frac{d\nu_2}{d\lambda} \Big/ \frac{d\nu_1}{d\lambda}\right)$ and the constant C_α is defined so that the test has significance level $\alpha\ (0<\alpha<1)$. If K is rather large in the sense $\nu_i(K) \longrightarrow \infty$, i=1,2 as $K \uparrow X$ we can employ the asymptotical normality of $\mu(hI_K)$ in order to define approximately C_α and the power of the test.

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