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CLOSURE OPERATIONS AND SEMIGROUPS OF QUOTIENTS

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Abstract: The semigroup of quotients of a monoid S with 0 is constructed by means of a closure operation c on the lattice of left ideals of S. The lattice of closure operations on L is isomorphic to the lattice of left quotient filters Σ on S. These closure operations enable one to describe Green's relation L on Q(S).

Key words: Semigroup of quotients, injective hull, closure operation.

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The discussion of semigroups of quotients has been modelled on the discussion of rings of quotients. One useful way of discussing rings of quotients was by the use of closure operations on the lattice of left ideals of a ring [2]. In this paper, we give a generalization of the work of Murdoch [2] on rings to semigroups. This generalization gives yet another way to describe the maximal semigroups of quotients of S. This new description leads to a characterization of Green's relation L_Q on Q(S) and shows the connection between L_S on S and L_Q on Q(S).

A nonempty collection Σ of left ideals of S is a <u>left quotient filter</u> if

(σ1) $A \subseteq B$, $A \in \Sigma \Rightarrow B \in \Sigma$

(σ2) $f \in \mathrm{Hom}\ (A,S); A,B \in \Sigma \Rightarrow f^{-1}(B) \in \Sigma$

(σ3) if $A \in \Sigma$, $Ba^{-1} = \{s \in S \mid sa \in B\} \in \Sigma$ for all $a \in A$, then $B \in \Sigma$.

In [1] we showed that each left quotient filter Σ determines a torsion congruence τ_M on each left S-system ${}_SM$ by $\tau_M = \{(m,n) \mid am = an$ for all $a \in A$, $A \in \Sigma\}$. Then ${}_SM$ is torsion free if the 0-class of $\tau_M = (0)$, torsion if $\tau_M = M \times M$, and strongly torsion free if $\tau_M =$ id. Given a strongly torsion free ${}_SM$, we constructed the Σ-injective hull of ${}_SM$, denoted by $E_\Sigma(M)$, as $B = \bigcup_{A \in \Sigma} \mathrm{Hom}\ (A,M)$, $\Theta = \{(f,g) \in B \times B :$ $: fa = ga$ for all a in some $A \in \Sigma\}$, and $E_\Sigma(M) = B/\Theta$. Then $M \subseteq E_\Sigma(M)$ and $E_\Sigma(M)$ is contained in the injective hull $E(M)$ of M. Moreover, $E_\Sigma(S)$ is a semigroup and is called the semigroup of quotients of S, denoted by $Q_\Sigma(S)$. $E_\Sigma(M)$ is Σ-injective in the sense that if A and B are S-systems with $Ab^{-1} \in \Sigma$ for all $b \in B$, and if $f \in \mathrm{Hom}\ (A, E_\Sigma(M))$, then there is $\bar{f} \in \mathrm{Hom}\ (B, E_\Sigma(M))$ with $\bar{f}(a) = f(a)$ for all $a \in A$. We refer the reader to [1] for the remaining definitions and for a more complete discussion of torsion theories.

1. *Closure Operations and Quotient Filters*. Let L be the collection of left ideals of S. A *closure operation* is a mapping $A \longrightarrow A^c$ of L into itself satisfying

(1) $A \subseteq A^c$

(2) $A \subseteq B \Rightarrow A^c \subseteq B^c$

(3) $A^{cc} = A^c$

(4) If $f \in \mathrm{Hom}_S(S,S)$, then $f^{-1}(A^c) = (f^{-1}A)^c$.

An element is *c-closed* if $A^c = A$, and is *c-dense* if

$A^c = S$. The intersection of any collection of c-closed elements is c-closed and so the closure A^c of A is the intersection of all c-closed elements containing A. Thus the closure operation is completely determined by the lattice L^c of closed elements of L, and L^c is an *inset* in the sense that $S \in L^c$, $L^c \subseteq L$ and L^c is closed under complete intersections. Conversely, each inset determines a unique closure operation on L.

Given a left quotient filter Σ on S, let $A \in L$ and set $A^c = \{s \in S \mid As^{-1} \in \Sigma\}$. Then the mapping $A \to A^c$ is a closure operation on L. Note that (2) follows from (σ1), (3) follows from (σ3) and (4) follows from (σ2).

Conversely, given a closure operation $A \to A^c$ on L, let $\Sigma = \{A \in L \mid A^c = S\}$.

Proposition: Σ is a left quotient filter.

Proof: Condition (σ 1) follows from (2). Condition (σ 2) follows immediately from (4). In order to verify (σ 3), let $A \in \Sigma$ and $Ba^{-1} \in \Sigma$ for all $a \in A$. Then $\bigcup_{a \in A} (Ba^{-1})a \subseteq B$, and so $[\bigcup_{a \in A} (Ba^{-1})a]^c \supseteq \bigcup_{a \in A} [(Ba^{-1})a]^c$. Define $f_a : S \to S$ by $f(s) = sa$. Then by (3), $(Ba^{-1})^c a = Sa \subseteq (Ba^{-1}a)^c$ so $\bigcup_{a \in A} Sa = A \subseteq \subseteq \bigcup (Ba^{-1}a)^c \subseteq [\bigcup (Ba^{-1})a]^c \subseteq B^c$ so $A^c = S \subseteq B^c$ and $B \in \Sigma$.

Given a left quotient filter Σ, we obtain a closure operation $A \to A^c$ and from this a left quotient filter Σ_c.

Proposition: $\Sigma = \Sigma_c$.

Proof: Let $A \in \Sigma_c$, then $A^c = S$ so for all $s \in S$, there is $B_s \in \Sigma$ with $B_s s \subseteq A$. Thus $S \in \Sigma$ and $As^{-1} \in \Sigma$ for all $s \in S$ implies $A \in \Sigma$ by (σ3). Conversely, $A \in \Sigma$ implies $As^{-1} \in \Sigma$ for all $s \in S$ so $A^c = S$ or $A \in \Sigma_c$.

Conversely, let $(\ )^c$ be a closure operation on L, and let $(\ )^{c_1}$ be the closure operation given by Σ_c.

Proposition: $c = c_1$.

Proof: We have $y \in B^{c_1}$ iff $Sy \subseteq B^{c_1}$ iff $S \subseteq B^{c_1}y^{-1} = (By^{-1})^{c_1}$ iff $(By^{-1})^{c_1} = S$ iff $(By^{-1})^c = S$ iff $B^c y^{-1} = S$ iff $y \in B^c$.

Define an order on the class of closure operations on L by $a \leq b$ iff $L^a \subseteq L^b$. Thus if $a \leq b$, for $A \in L$ we have $A^b \subseteq A^a$. Now $L^a \cap L^b$ is an inset and so yields a closure operation $a \vee b$. Likewise $L^a \cup L^b$, the set consisting of all elements of the form $A \cap B$ where $A \in L^a$ and $B \in L^b$, is also an inset and so yields a closure operation a b. This gives the class of closure operations on L the structure of a complete lattice.

Combining these results we have the following:

Theorem: The collection of left quotient filters on S is in one-to-one correspondence with the lattice of closure operations on L by the map $\Sigma \to (\ )^c$. Moreover, this map satisfies $\Sigma_{a \vee b} = \Sigma_a \cap \Sigma_b$.

We remark that each closure operation can be extended to the category of left S-systems by defining for a left S-system M, E(M) the injective hull of M, and

$$M^c = \{ e \in E(M) : Me^{-1} \in \Sigma \}.$$

Thus to each closure operation c there corresponds a torsion theory $(\overline{T}, \overline{F})$ given by $\overline{T} = \{A \mid 0^c = A\}$ and $\overline{F} = \{B \mid 0^c = 0\}$, where by $0^c = A$ we mean that for $a \in A$ there is $T \in \Sigma$ with $Ta = 0$. We refer the interested reader to [1] for a discussion of torsion theories.

2. <u>Closure Operations and Injective Hulls</u>. Let c be a closure operation on L and let Σ be its corresponding left quotient filter. Let ${}_S M$ be a strongly torsion free left S-system. Let $E(M) = E$ be the injective hull of M. Let T be the closure of M in E, that is, $T = \{ e \in E \mid Ae \in M \text{ for some } A \in \Sigma \}$. Now M is $\cap$-large in T since $0^c = 0$.

<u>Lemma</u>: T is Σ-injective.

<u>Proof</u>: Let

$$\begin{tikzcd}
A \arrow[r, phantom, "\subseteq'"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
T \arrow[r, phantom, "\subseteq"] & E
\end{tikzcd}$$

where $A \subseteq' B$ means that for all $b \in B$, there is $C \in \Sigma$ with $Cb \subseteq A$. Then if $g(b) \notin T$, there is $C \in \Sigma$ with $Cb \subseteq A$ so $Cg(b) = g(Cb) \subseteq f(A)$ T so $g(b) \in T^c = M^{cc} = M^c = T$.

Thus T is a Σ-injective $\cap$-large extension of M. T is unique up to isomorphism over M for if F is another Σ-injective $\cap$-large extension of M, then we have the commutative diagram

$$\begin{tikzcd}
M \arrow[r, phantom, "\subseteq"] \arrow[d, equal] & T \arrow[d, shift left, "g"] \\
M \arrow[r, phantom, "\subseteq"] & F \arrow[u, shift left, "h"]
\end{tikzcd}$$

Then g = h if we can show the following

<u>Lemma</u>: T is strongly Σ-injective.

Proof: Let

$$\begin{tikzcd}
A \arrow[r, phantom, "\subseteq'"] \arrow[d, "f"'] & B \arrow[d, shift right, "h"'] \arrow[d, shift left, "g"] \\
T \arrow[r, phantom, "\subseteq"] & E
\end{tikzcd}$$

be as before and suppose for some $b \in B$, $g(b) \neq h(b)$. Then for $C \in \Sigma$ with $Cb \subseteq A$, $Cg(b) = Ch(b)$. Thus $(g(b),h(b)) \in \tau_T = id$ so $g(b) = h(b)$.

In [1], we showed that when $M = S$, T is a semigroup and is called the Σ-semigroup of quotients of S and is denoted by $Q_\Sigma(S)$. We next examine whether the subsemigroups of $Q(S)$, the maximal Utumi semigroup of quotients of S, all occur as $Q_\Sigma(S)$ for some left quotient filter Σ. Since each left quotient filter is determined by a unique closure operation c, we will discuss subrings of $Q(S)$ corresponding to bilateral closure operations on L. A closure operation c on L is <u>bilateral</u> if A^c is a two sided ideal.

Recall the construction of $Q(S)$ as $Q(S) = B/\Theta$ where $B = \bigcup_{A \in S}\Delta \text{ Hom }(A,S)$ with operation fg as composition where $f : I_f \rightarrow S$, $g : I_g \rightarrow S$ then $fg : I_{fg} \rightarrow S$ where $I_{fg} = I_f \cap g(I_g)$, and $f \,\Theta\, g$ iff $f(x) = g(x)$ for all x in some $A \in \Sigma$. Let $B_c = \{f \in B \mid fJ \subset J^c$ for all left ideals $J \subseteq I_g\}$. Then B_c is a subsemigroup of B for if $J \subset I_{fg}$, $f_gJ \subseteq f(I_f \cap J^c) \subseteq (I_f \cap J^c)^c \subseteq S \cap J^c = J^c$. (Note that this requires only that c is a closure operation.) Set $Q_c(S) = B_c/\Theta_c$ where Θ_c is the restriction of Θ to B_c. Then $Q_c(S)$ is a subsemigroup of $Q(S)$. Moreover, if c is bilateral, then for $s \in S$, $I_s = S$ and $As \subseteq A^c s \subseteq A^c$ so $Q_c(S) \supseteq S$. We gather these results in the followin

<u>Theorem</u>: If c is a closure operation on L, then $Q_c(S)$ is a subsemigroup of $Q(S)$. If c is a bilateral closure operation on L, then $S \subseteq Q_c(S) \subseteq Q(S)$.

This construction enables one to prove the following results due to Murdoch [2] for rings.

Proposition: For each bilateral closure c on L, there is a unique bilateral closure $\bar{c}$ such that $Q_c(S) = Q_{\bar{c}}(S)$ and the mapping $c \rightarrow \bar{c}$ is a closure operation on the lattice of bilateral closure operations on L. $\bar{c}$ is called a *maximal bilateral closure operation* on L.

Let $(M, \cap, \cup^*, \leq)$ be the complete lattice of maximal bilateral closure operations on L with $a \leq b$ as before and $\cup^* c_\alpha = \overline{\cup c_\alpha}$. A subsemigroup T of Q(S) is a *closure subsemigroup* if $T = Q_c(S)$ for some (maximal) bilateral closure operation on L.

Proposition: The lattice $(M, \cap, \cup^*, \leq)$ is anti-isomorphic to the lattice C of closure subrings of Q by the mapping $c \rightarrow Q_c(S)$.

In order to construct $\bar{c}$ for a bilateral closure c, we let $J^q = \bigcup_{f \in B_c} f(J \cap I_f)$. Then q^* is a bilateral closure operation and $q^* = \bar{c}$ if q^* is the closure operation given by $\{J \in L \mid J^q = J\}$.

Next let R be any subset of Q containing S and construct $J^q = \bigcup_{a \in R} \alpha(J \cap I_\alpha)$ and so q^*_R.

Theorem: Let $Q_c(S)$ be the minimal closure subsemigroup of Q(S) containing R, then $Q_{q^*_R}(S) = Q_c(S)$ and $q^*_R = \bar{c}$.

3. **Closure Operations and Green's Relation** L. Let $Q_\Sigma(S)$ be the semigroup of quotients of S with respect to Σ. Note that $S \subseteq Q_\Sigma(S)$ iff $\tau_S = \{(s,t) \mid as = at \text{ for a in some } A \in \Sigma\} = \text{id}$. Green's relation L_S on S is given by $L_S = \{(x,y) \mid Sx = Sy\}$. Define the relation L_Σ on S by $L_\Sigma = \{(x,y) \mid Ax = Ay \text{ for some } A \in \Sigma\}$. Note that $\tau_S \subseteq L_\Sigma$ but in general they

are unequal.

Recall [1] that Σ has property (T) if every $Q_{\Sigma}(S)$ system is strongly torsion free and this is equivalent to $Q_{\Sigma}(S)i(A) = Q_{\Sigma}(S)$ for all $A \in \Sigma$ where $i(A)$ is the image of A in $Q_{\Sigma}(S)$ under the canonical mapping of S into $Q_{\Sigma}(S)$. Hence if Σ has property (T) and $(s,t) \in L_{\Sigma}$, then $As = At$ so $i(A)i(s) = i(A)i(t)$ in $Q_{\Sigma}(S)$ so $Q_{\Sigma}(S)i(s) = Q_{\Sigma}(S)i(t)$ and $(i \times i) L_{\Sigma} \subseteq L_Q$. Moreover, since $S \in \Sigma$, $L_S \subseteq L_{\Sigma}$.

Proposition: If Σ has property (T), then $(i \times i) L_{\Sigma} \subseteq L_Q \cap (i(s) \times i(s))$. In any case $L_S \subseteq L_{\Sigma}$.

Let $s \in S$, and suppose S is strongly torsion free so that $S \subseteq Q_{\Sigma}(S)$. Then $[Q_{\Sigma}(S)s \cap S]^c = (Ss)^c$ for if $qs = t \in S$, then there is $B \in \Sigma$ with $Bq \subseteq S$ so $Bqs \subseteq Ss$ so $qs \in (Ss)^c$. Now suppose $(s,t) \in L_Q \cap S \times S$, then $Q_{\Sigma}(S)s = Q_{\Sigma}(S)t$ so $(Ss)^c = (St)^c$.

Now suppose Σ has property (T). Then if $(Ss)^c = (St)^c$, there are $A, B \in \Sigma$ with $As \subseteq St$ and $Bt \subseteq Ss$. Thus $Q_{\Sigma}(S)As \subseteq Q_{\Sigma}(S)St$ or $Qs \subseteq Qt$ and $Qt \subseteq Qs$ so $(s,t) \in L_Q \cap S \times S$.

This same argument suffices to show:

Proposition: Let S be strongly torsion free and Σ have property (T), then

$$L_Q = \{(\alpha, \beta)\ [(S\alpha^{-1})\alpha]^c = [(S\beta^{-1})\beta]^c\}.$$

Finally let $\Sigma = \langle M \rangle$ where M is an idempotent two sided ideal of S. If $(\alpha, \beta) \in L_Q$, then $M\alpha \subseteq S\beta$ and by (Q3), $M\beta \subseteq M\alpha$. Since the argument is symmetric, $L_Q = \{(\alpha, \beta) \mid M\alpha = M\beta\} = L_{\Sigma}$.

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