COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
20,3 (1979)

SOME BAIRE CATEGORY TYPE THEOREMS FOR $U(\omega_1)$

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Abstract: It is shown that if ${}^{\omega}\omega$ has an ω_1-scale, then $U(\omega_1)$ can be covered by ω_1 G_δ closed and nowhere dense subsets of $U(\omega_1)$ and that the union of countably many of them is dense in $U(\omega_1)$. On the other hand, we show that under MA+ $\neg$CH, the union of countably many G_δ, closed and nowhere dense subsets of $U(\omega_1)$ is nowhere dense in $U(\omega_1)$. For these purposes we use the notion of κ-matrices on ω_1.

Key words and phrases: Ultrafilter, uniform ultrafilter, matrix, scale.

AMS: Primary 04A30, 06A40, 54D40, 54D35
Secondary 54F65, 54G05

In this note we consider families consisting of G closed and nowhere dense subsets of $U(\omega_1)$. We are mainly interested in the question, what cardinalities have such families, as above, which cover $U(\omega_1)$ or have a dense union. Some results in this direction are obtained. For example, it is shown (Theorem 2) that if ${}^{\omega}\omega$ has an ω_1-scale, then such a family of cardinality ω_1 exists which covers $U(\omega_1)$ and, in addition, it contains a countable subfamily with a dense union. The same conclusions have been obtained by Balcar and Vopěnka [BV] when

$2^{\omega_1} = \omega_2$ holds, however, without possibility to get G_δ-sets. Our result also shows that if ${}^{\omega}\omega$ has an ω_1-scale, then the Novák number of $U(\omega_1)$, $n(U(\omega_1))$, is $\leq \omega_1$. Recall [KS] that the <u>Novák number</u> of a dense in itself topological space X, n(X), is the minimal cardinality of a family consisting of nowhere dense sets covering the whole space. For the short history concerning the Novák number of various topological spaces, we refer to [BPS].

The existence of families consisting of G_δ closed and nowhere dense subsets of $U(\omega_1)$ is closely related to the existence of κ-matrices on ω_1, as is shown in Theorem 4, and the existence of κ-matrices on ω_1 for $\kappa \geq \omega_1$ is related to the question whether $\beta\omega_1 - \omega_1$ is homeomorphic to $\beta\omega - \omega$ (Theorem 6).

All of the above results are independent of the ZFC axioms since if Q holds, then the union of countably many G_δ closed and nowhere dense subsets of $U(\omega_1)$ is nowhere dense in $U(\omega_1)$ (Theorem 8).

<u>Conventions and notations</u>. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Cardinals carry the discrete topology. If A, B are sets, then ${}^{A}B$ is the set of all functions from A into B. If $\varphi, \psi \in {}^{\omega}\omega$, then $\varphi \nleq \psi$ means that $|\{n: \varphi(n) \geq \psi(n)\}| < \omega$. A subset $F \subset {}^{\omega}\omega$ is <u>dominant</u> if for every $\varphi \in {}^{\omega}\omega$ there is a $\psi \in F$ such that $\varphi \nleq \psi$. A <u>scale</u> is a well ordered by $\nleq$, increasing dominating family. If κ is a cardinal and $A, B \subset \kappa$, then A and B are <u>almost disjoint</u> if $|A| = \kappa = |B|$ and $|A \cap B| < \kappa$. We denote by $U(\omega_1)$ the space of uniform ultrafilters on ω_1.

Results. We begin from the following simple

Lemma 1. A set $F \subset U(\omega_1)$ is G_δ closed and nowhere dense in $U(\omega_1)$ iff for any sets $A_n \subset \omega_1$, $n < \omega$, such that $F = \bigcap\{cl_{\beta\omega_1} A_n : n < \omega\} \cap U(\omega_1)$ there is $|\bigcap\{A_n : n < \omega\}| \le \omega$ iff there are sets $B_n \subset \omega_1$ such that $F = \bigcap\{cl_{\beta\omega_1} B_n : n < \omega\} \cap U(\omega_1)$, $B_1 \supset B_2 \supset \ldots$ and $\bigcap\{B_n : n < \omega\} = \emptyset$.

Theorem 2. If ${}^\omega\omega$ has an ω_1-scale, then $U(\omega_1)$ can be covered by ω_1 G_δ closed and nowhere dense subsets of $U(\omega_1)$. In particular, if ${}^\omega\omega$ has an ω_1-scale, then $n(U(\omega_1)) = \omega_1$.

Proof. Let $\{\varphi_\alpha : \alpha < \omega_1\}$ be an ω_1-scale in ${}^\omega\omega$. For each $n, m < \omega$ we set $A_n^m = \{\alpha : \varphi_\alpha(n) \le m\}$. Observe that:

(0) if $m < k < \omega$ and $n < \omega$, then $A_n^m \subset A_n^k$,

(i) $\bigcup\{A_n^m : m < \omega\} = \omega_1$ for each $n < \omega$,

(ii) for each infinite $s \subset \omega$ and $\psi \in {}^s\omega$, $|\bigcap\{A_n^{\psi(n)} : n \in s\}| \le \omega$.

The properties of A_n^m's stated in (0) and (i) are obvious. For the proof of (ii) let us assume on the contrary that $|\bigcap\{A_n^{\psi(n)} : n \in s\}| > \omega$ for some infinite $s \subset \omega$ and $\psi \in {}^s\omega$. There exists an $\alpha < \omega_1$ such that $\varphi_\alpha | s \not\le^* \psi | s$. Since $\bigcap\{A_n^{\psi(n)} : n \in s\}$ is uncountable, there exists a $\beta \in \bigcap\{A_n^{\psi(n)} : n \in s\}$ such that $\omega_1 > \beta > \alpha$. Since $\{\varphi_\alpha : \alpha < \omega_1\}$ is a scale, $\varphi_\beta \ge^* \varphi_\alpha$. Hence there is an $n \in s$ such that $\varphi_\beta(n) > \psi(n)$. But this means that $\beta \notin A_n^{\psi(n)}$; a contradiction.

Now define the sets F_n and E_n in the following way:

$F_n = \{\xi \in U(\omega_1) : A_n^m \notin \xi$ for each $m < \omega\}$ and

$E_n^\alpha = \{\xi \in U(\omega_1) : A_m^{\varphi_\alpha(m)} \in \xi$ for each $m \ge n\}$.

In the topological language, $F_n = \bigcap\{cl_{\beta\omega_1}(\omega_1 - A_n^m) : m < \omega\}$

and $E_n^{\alpha} = \bigcap \{ cl_{\beta\omega_1} A_m^{\varphi_\alpha(m)} : m \geq n \}$. Of course F_n as well as E_n^{α} are G_δ closed subsets of $U(\omega_1)$. From (i) and Lemma 1 it follows that F_n is nowhere dense in $U(\omega_1)$ for each $n < \omega$, and from (ii) and Lemma 1 it follows that E_n^{α} is nowhere dense in $U(\omega_1)$ for each $n < \omega$ and $\alpha < \omega_1$. It remains to show that $\bigcup \{ F_n : n < \omega \} \cup \bigcup \{ E_n^{\alpha} : n < \omega , \alpha < \omega_1 \} = U(\omega_1)$. For this, let $\xi \in U(\omega_1)$ be such that $\xi \notin \bigcup \{ F_n : n < \omega \}$. From (0) it follows that for each $n < \omega$ there exists $\psi(n) < \omega$ such that $A_n^{\psi(n)} \in \xi$. Let $\alpha < \omega_1$ be such that $\varphi_\alpha \geq_* \psi$. This means that there exists an $m < \omega$ such that $\varphi_\alpha(n) > \psi(n)$ for each $n \geq m$. Hence $\xi \in E_m^{\alpha}$.

The above theorem is related to a result by Balcar and Vopěnka [BV] who proved that if $2^{\omega_1} = \omega_2$, then $n(U(\omega_1)) = \omega_1$. However, the following consistency results are known:

(${}^{\omega}\omega$ has an ω_1-scale + $2^{\omega_1} = 2^{\omega} + 2^{\omega}$ arbitrarily large) [H],

($\neg\, {}^{\omega}\omega$ has an ω_1-scale + $2^{\omega_1} = \omega_2$) (a model for Martin's axiom + $2^{\omega} = \omega_2$ [MS]),

(${}^{\omega}\omega$ has an ω_1-scale + $2^{\omega_1} = \omega_2$) (a model for GCH).

In the proof of the Theorem 2, we have constructed a matrix $\{A_n^m : m,n < \omega\}$ satisfying conditions (0),(i,)(ii). Now we generalize this notion by saying that a matrix $\{A_\alpha^n : n < \omega , \alpha < \kappa\}$ of subsets of ω_1 is a κ-matrix on ω_1 if the following hold:

(0) if $m < n$ and $\alpha < \kappa$, then $A_\alpha^m \subset A_\alpha^n$,

(i) $\bigcup \{ A_\alpha^n : n < \omega \} = \omega_1$ for each $\alpha < \kappa$,

(ii) for each infinite $s \subset \kappa$ and $\psi \in {}^{s}\omega$, $| \bigcap \{ A^{\psi(\alpha)} : \alpha \in s \} | \leq \omega$.

Thus we have shown

Proposition 3. If ${}^{\omega}\omega$ has an ω_1-scale, then there exists an ω-matrix on ω_1.

Now we shall give a topological reformulation of the existence of κ-matrices on ω_1.

Theorem 4. A κ-matrix on ω_1 exists iff there exists a family consisting of at least κ G_δ closed and nowhere dense subsets of $U(\omega_1)$ such that each union of infinitely many of them is dense in $U(\omega_1)$.

Proof. Assume $\{A^n_\alpha : n < \omega, \alpha < \kappa\}$ is a κ-matrix on ω_1. For $\alpha < \kappa$ we put $F_\alpha = \{\xi \in U(\omega_1) : A^n_\alpha \notin \xi$ for each $n < \omega\}$. Obviously, each F_α is a G_δ closed and nowhere dense subset of $U(\omega_1)$, in virtue of Lemma 1 and (i). Choose infinitely many of them, say $F_{\alpha_1}, F_{\alpha_2}, \dots$ and assume on the contrary that $F_{\alpha_1} \cup \cup F_{\alpha_2} \cup \dots$ is not dense in $U(\omega_1)$. This means that there exists an uncountable set $B \subset \omega_1$ such that $cl_{\beta\omega_1} B \cap F_{\alpha_n} = \emptyset$ for each $n < \omega$. Hence, by (0) and (i), for each $n < \omega$ there exists a $\psi_n < \omega$ such that $|B - A^{\psi_n}_{\alpha_n}| \leq \omega$. Hence B contains an uncountable subset C such that $C \subset A^{\psi_n}_{\alpha_n}$ for each $n < \omega$. But then, for some infinite set $s = \{\alpha_1, \alpha_2, \dots\}$ contained in κ and a $\psi \in s_\omega$ given by $\psi(\alpha_n) = \psi_n$, we have $|\bigcap\{A^{\psi(\alpha)} : \alpha \in s\}| \geq \geq |C| = \omega_1$, which contradicts (ii).

Let F_α, $\alpha < \kappa$, be G_δ closed and nowhere dense subsets of $U(\omega_1)$ such that each union of infinitely many of them is dense in $U(\omega_1)$. By Lemma 1, for each $\alpha < \kappa$ there are sets B^n_α, $n < \omega$, such that $F_\alpha = \bigcap\{cl_{\beta\omega_1} B^n_\alpha \cap U(\omega_1) : n < \omega\}$, $B^1_\alpha \supset \supset B^2_\alpha \supset \dots$ and $\bigcap\{B^n_\alpha : n < \omega\} = \emptyset$. Setting $A^n_\alpha = \omega_1 - B^n_\alpha$ we see that the matrix $\{A^n_\alpha : n < \omega, \alpha < \kappa\}$ fulfils conditions (0) and (i). We verify (ii). Choose an arbitrary infinite set $s \subset \kappa$

and $\psi \in {}^s\omega$. By the assumption, $\bigcup \{ F_\alpha : \alpha \in s\}$ is dense in $U(\omega_1)$, so that $\bigcap \{ cl_{\beta\omega_1} A^{\psi(\alpha)} \cap U(\omega_1) : \alpha \in s\}$ is nowhere dense in $U(\omega_1)$. Hence, by Lemma 1, $|\bigcap \{ A_\alpha^{\psi(\alpha)} : \alpha \in s\}| \leq \leq \omega$.

Corollary 5. An ω-matrix on ω_1 exists iff there is a countable family F consisting of G_δ closed and nowhere dense subsets of $U(\omega_1)$ such that $\bigcup F$ is dense in $U(\omega_1)$.

Proof. If $F = \{E_n : n < \omega\}$, then letting $F_1 = E_1$ and $F_n = E_1 \cup E_2 \cup \ldots \cup E_n$ for $1 < n < \omega$, we see that each F_n is a G_δ closed and nowhere dense subset of $U(\omega_1)$ such that each union of infinitely many of them is dense in $U(\omega_1)$, since it is equal to $\bigcup F$.

The above topological equivalence of the existence of κ-matrices on ω_1 seems to be rather pathological, for $\kappa \geq \geq \omega_1$. For example, it cannot happen in topological spaces which have a pseudobase of cardinality less than κ. However, we have

Theorem 6. If $\beta\omega_1 - \omega_1$ is homeomorphic to $\beta\omega - \omega$ and there exists an almost disjoint family on ω_1 of cardinality κ, then there exists a κ-matrix on ω_1.

Proof. Decompose ω_1 into ω_1 disjoint subsets B_α of cardinality ω_1, say $B_\alpha = \{ b_\alpha^\beta : \beta < \omega_1\}$. Let $F = \{f_\xi : \xi < \kappa\}$ be a family consisting of almost disjoint subsets of ω_1. Let φ_ξ be an isomorphism between ω_1 and a well ordered set f_ξ. Then we put $C_\xi = \{ b_\alpha^{\varphi_\xi(\alpha)} : \alpha < \omega_1\}$. Note that sets C_ξ defined in such a way are also almost disjoint and $|C_\xi \cap B_\alpha| = 1$ for each $\xi < \kappa$ and $\alpha < \omega_1$.

Let ϕ be a Boolean isomorphism between the Boolean algebras $P(\omega_1)$/mod fin and $P(\omega)$/mod fin. Choose $B'_\alpha \in \phi([B_\alpha])$ and $C'_\xi \in \phi([C_\xi])$. Then we define $A^n_\xi = \{\alpha : B'_\alpha \cap C'_\xi \subset n\}$. The matrix $\{A^n_\xi : \xi < \kappa, n < \omega\}$ is a κ-matrix on ω_1. To see this, observe that conditions (0) and (i) follow from the fact that B'_α and C'_ξ are almost disjoint subsets of ω, for each $\alpha < \omega_1$ and $\xi < \kappa$. We verify (ii). Let infinite $s \subset \kappa$ and $\psi \in {}^s\omega$ be given. Assume on the contrary that $|\bigcap\{A^{\psi(\xi)} : \xi \in s\}| > \omega$. Without loss of generality we may assume that s is countable. Let $D' = \bigcup\{C'_\xi - \psi(\xi) : \xi \in s\}$ and choose $D \in \phi^{-1}([D'])$. Since $|C'_\xi - D'| < \omega$ for each $\xi \in s$, $|C_\xi - D| < \omega$ for each $\xi \in s$. Since s is countable, there is a $\beta < \omega_1$ such that $C_\xi - \bigcup\{B_\alpha : \alpha < \beta\} \subset D$. Since the sets C_ξ are almost disjoint, there is a $\gamma < \omega_1$ such that the sets $C_\xi - \bigcup\{B_\alpha : \alpha < \gamma\}$ are disjoint for each $\xi \in s$. Consequently, $|\bigcup\{C_\xi : \xi \in s\} \cap B_\alpha| = \omega$ for each $\alpha > \gamma$. Choose $\eta \in \bigcap\{A^{\psi(\xi)}_\xi : \xi \in s\}$ such that $\eta > \beta$ and $\eta > \gamma$. Then $|B_\eta \cap D| = \omega$ and therefore $|B'_\eta \cap D'| = \omega$, too. Thus $B'_\eta \cap C'_\xi \not\subset \psi(\xi)$ for infinitely many ξ. Hence $\eta \notin \bigcap\{A^{\psi(\xi)} : \xi \in s\}$; a contradiction.

Since there exists always an almost disjoint family on ω_1 of cardinality ω_2, we have

<u>Corollary 7</u>. If $\beta\omega_1 - \omega_1$ is homeomorphic to $\beta\omega - \omega$, then there exists an ω_2-matrix on ω_1.

The problem to distinguish topologically the spaces $\beta\omega_1 - \omega_1$ and $\beta\omega - \omega$ is not yet solved; for partial solutions see [F], [BF].

Some theorems above show what kinds of conditions allow to get the existence of some κ-matrices on ω_1. The next theorem refutes such a possibility.

Q means that if $F \subset {}^{\omega}\omega$ and $|F| \leq \omega_1$, then there is a $\psi \in {}^{\omega}\omega$ such that $\varphi \lneq_* \psi$ for each $\varphi \in F$.

<u>Theorem 8</u>. If Q, then there is no ω-matrix on ω_1.

Proof. Assume otherwise and let $\{A_n^m : n,m < \omega\}$ be an ω-matrix on ω_1. For $\varphi \in {}^{\omega}\omega$ we let $a^{\varphi} = \sup\{b_n^{\varphi} : n < \omega\}$, where $b_n^{\varphi} = \sup \bigcap \{A_k^{\varphi(k)} : k \geq n\}$. Since $\{A_n^m : n,m < \omega\}$ is an ω-matrix on ω_1, $a^{\varphi} < \omega_1$ for each $\varphi \in {}^{\omega}\omega$. Now, we claim that for each $\alpha < \omega_1$ there is a $\varphi_\alpha \in {}^{\omega}\omega$ such that $a^{\varphi_\alpha} \geq \alpha$. To see this, we note that from condition (i) for κ-matrices it follows that for each $n < \omega$ there exists $\varphi_n < \omega$ such that $\alpha \in A_n^{\varphi_n}$. So, taking φ_α such that $\varphi_\alpha(n) = \varphi_n$, we have $a^{\varphi_\alpha} \geq \alpha$. By Q, there exists a $\psi \in {}^{\omega}\omega$ such that $\varphi_\alpha \lneq_* \psi$ for each $\alpha < \omega_1$. Let $\beta < \omega_1$. Since $\varphi_\beta \lneq_* \psi$, there exists an $n < \omega$ such that $\varphi_\beta(k) < \psi(k)$ for $k \geq n$. Then, by (0) for κ-matrices, $A_k^{\varphi_\beta(k)} \subset A_k^{\psi(k)}$ for $k \geq n$. Hence $\bigcap\{A_k^{\varphi_\beta(k)} : k \geq n\} \subset \bigcap\{A_k^{\psi(k)} : k \geq n\}$, and therefore $b_m^{\varphi_\beta} \leq b_m^{\psi}$, for each $m \geq n$. In consequence, $\beta \leq a^{\varphi_\beta} = \sup\{b_n^{\varphi_\beta} : n < \omega\} \leq \sup\{b_n^{\psi} : n < \omega\} = a^{\psi}$. Hence $a^{\psi} = \omega_1$; a contradiction.

It is well known that Martin's axiom + $\neg$CH implies Q ([MS]). So we have

<u>Corollary 9</u> (MA + $\neg$CH). If F is a countable family consisting of G closed and nowhere dense subsets of $U(\omega_1)$, then $\bigcup F$ is nowhere dense in $U(\omega_1)$.

If F is a countable family consisting of disjoint closed and nowhere dense subsets of $U(\omega_1)$, then $\bigcup F$ is nowhere dense in $U(\omega_1)$.

Proof. Assume otherwise. Then, by Corollary 5, some uncountable subset of ω_1 would have an ω-matrix. But this contradicts Theorem 8.

The second part of the corollary follows immediately from the first part.

It may be worthwhile to point out that the assumptions on the family F in Corollary 9 are essential, since Balcar and Vopěnka [BV] showed that if $2^{\omega_1} = \omega_2$, then there exists a countable family F' consisting of closed and nowhere dense subsets of $U(\omega_1)$ such that $\bigcup F'$ is dense in $U(\omega_1)$. Also $2^{\omega_1} = \omega_2$ is consistent with MA + $\neg$ CH.

<u>Question</u>. Does the existence of κ-matrices on ω_1, for $\kappa \geq \omega_1$, be consistent with ZFC?

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(Oblatum 5.5. 1979)