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GENERALIZED PROJECTIVITY – II

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Abstract: Recently in [11] the (r,i,s,j)-projectivity (i.e. the projectivity with respect to two preradicals r and s) has been investigated. In many cases the (r,i,s,j)-projectivity is reduced to the (1,t)-projectivity for some preradical t. It is shown that a module P is (1,r)-projective if and only if P/ch(r)(P) is projective in R/r(R)-mod. In § 2 we shall show that the concepts of (1,r)-projectivity and the strongly M-projectivity which is studied by K. Varadarajan in [18] are the same. Further, in the study (r,2)-projectivity, where r is an idempotent preradical and $\tilde{r}$ is pseudohereditary, r can be replaced by a hereditary radical. § 3 is devoted to the study of (r,i,s,j)-quasiprojective modules. Some of these results are motivated by J.S. Golan's paper [8] on quasiprojective modules.

Key words: Generalized projectivity, generalized M-projectivity, generalized quasiprojectivity, preradicals.

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By R-mod we understand the category of all unitary left modules over an associative ring with unit element. The injective hull of a module M will be denoted by E(M), the direct product (sum) by $\prod_{i\in I} M_i$ ($\sum_{i\in I}^{\oplus} M_i$).

First, several basic definitions from the theory of preradicals (for details see [1],[2],[3],[5] and [12]).

A preradical r for R-mod is a subfunctor of the identity functor, i.e. r assigns to each module M its submodule r(M) in such a way that every homomorphism of M into N induces a homomorphism of r(M) into r(N) by restriction. A module M is r-torsion if r(M) = M and r-torsionfree if r(M) = 0. We shall denote by $\mathcal{T}_r$ ($\mathcal{F}_r$) the class of all r-torsion (r-torsionfree) modules.

A preradical r is said to be

- idempotent if r(r(M)) = r(M) for every module M,
- a radical if r(M/r(M)) = 0 for every module M,
- hereditary if $r(N) = N \cap r(M)$ for every submodule N of a module M,
- cohereditary if r(M/N) = (r(M) + N)/N for every submodule N of a module M,
- pseudohereditary if every submodule of r(R) is r-torsion,
- faithful if r(R) = 0.

We shall say that a module M splits in a preradical r if r(M) is a direct summand in M. If r and s are preradicals then we write $r \leq s$ if $r(M) \subseteq s(M)$ for all $M \in$ R-mod. The idempotent core $\bar{r}$ of a preradical r is defined by $\bar{r}(M) = \sum K$, where K runs through all r-torsion submodules K of M, and the radical closure $\tilde{r}$ is defined by $\tilde{r}(M) = \bigcap L$, where L runs through all submodules L of M with M/L r-torsionfree. Further, the hereditary closure h(r) is defined by $h(r)(M) = M \cap r(E(M))$ and the cohereditary core ch(r) by ch(r)(M) = r(R)M. For a preradical r and modules $N \subseteq M$ let us define $C_r(N:M)$ by $C_r(N:M)/N = r(M/N)$. Let r and s be two preradicals. A preradical t defined by $t(M) = C_s(r(M):M)$, $M \in$ R-mod, will be denoted by $r \Delta s$. For an arbitrary class of R-modules $\mathcal{A}$ we define $p^{\mathcal{A}}(N) = \bigcap \operatorname{Ker} f$, f ranging over all $f \in \operatorname{Hom}_R(N,M)$, $M \in \mathcal{A}$. As it is easy to see

p^{a} is a radical. Further, M is a pseudo-injective module iff $p^{\{M\}}$ is hereditary and M is a faithful module if and only if $p^{\{M\}}$ is faithful. Let $f:R\to S$ be a ring onto homomorphism and r be a preradical for R-mod. For all $M\in S$-mod let us define $f[r](M)= =S\cdot r({}_R M)$. Then $f[r]$ is a preradical for S-mod and $f[\bar{r}]=\overline{f[r]}$, $f[\tilde{r}]= =\widetilde{f[r]}$. Finally, the zero functor will be denoted by zer.

§ 1. (r,i,s,j)-projective modules. We start with some definitions which are introduced in [11]. Let s be a preradical for R-mod. An epimorphism $A\xrightarrow{h}B$ is said to be:

- (s,1)-codense if there exist $C\in R$-mod and $g:C\to A$ an epimorphism with $s(g^{-1}(\mathrm{Ker}\ h))\subseteq \mathrm{Ker}\ g$,
- (s,2)-codense if $s(\mathrm{Ker}\ h) = 0$,
- (s,3)-codense if $\mathrm{Ker}\ h\cap s(A) = 0$.

Further if $N\subseteq M$ is a submodule and $M\to M/N$ is a natural epimorphism which is (s,1)-codense, then we write $N\subseteq^{(s,1)}M$. Similarly $N\subseteq^{(s,2)}M$ $(N\subseteq^{(s,3)}M)$.

Let r,s be two preradicals, $i,j\in\{1,2,3\}$ and $M\in R$-mod. A module P is said to be (r,i,s,j,M)-projective if every diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow g & & \\ M & \xrightarrow[h]{} & N & \longrightarrow & 0 \end{array}$$

with exact row, $\mathrm{Ker}\ h\subseteq^{(r,i)}M$ and $h^{-1}(\mathrm{Im}\ g)\subseteq^{(s,j)}M$ can be completed to commutative one.

We say that a module P is (r,i,s,j)-projective if it is (r,i,s,j,M)-projective for all $M\in R$-mod.

A module P is said to be (r,i,s,j)-quasiprojective if it is (r,i,s,j,P)-projective.

A module P is said to be (r,i,M)-projective ((r,i)-(quasi) projective), if it is (r,i,zer,1,M)-projective ((r,i,zer,1)-(quasi) projective).

A module P is said to be (i,r,M)-projective ((i,r)-(quasi) projective), if it is (zer,1,r,i,M)-projective ((zer,1,r,i)-(quasi) projective).

As it is noted in [11] a module P is (r,i,s,j)-projective, iff it is (r,i,M)-projective for all $M \in$ R-mod with $M \leq^{(s,j)} M$, $i,j \in \{1,2,3\}$.

Let A,B be modules and let $\varphi: A \longrightarrow B$ be an epimorphism. A pair (A,φ) is said to be an (r,i,s,j,M)-projective ((r,i,s,j)-(quasi) projective) precover of the module B if A is (r,i,s,j,M)-projective ((r,i,s,j)-(quasi) projective), $A \xrightarrow{f} C \xrightarrow{g} B$ with $g \circ f = \varphi$, f,g epimorphisms and C (r,i,s,j,M)-projective ((r,i,s,j)-(quasi) projective) implies f is an isomorphism. An (r,i,s,j,M)-projective ((r,i,s,j)-(quasi) projective) precover (A,φ) which is a cover (i.e. $\mathrm{Ker}\,\varphi$ is superfluous in A) is said to be an (r,i,s,j,M)-projective ((r,i,s,j)-(quasi) projective) cover.

It is shown in [11] that (r,i,s,j,M)-projective ((r,i,s,j)-projective) cover of a module B exists whenever B has a projective cover.

<u>Proposition 1.1.</u> Let r,s be preradicals for R-mod, $j \in \{1,2\}$ and $P \in$ R-mod. Then

(i) if P is projective and $K \in \mathcal{T}_r$ then P/K is (r,1)-projective,

(ii) if P is (r,2,s,j)-projective and $K \in \mathcal{T}_{\tilde{r}}$ then P/K is (r,2,s,j)-projective.

(iii) if P is (r,3,s,j)-projective and $K \subseteq r(P)$ then P/K is (r,3,s,j)-projective.

<u>Proof</u>: Obvious.

<u>Proposition 1.2</u>. Let r,s be preradicals for R-mod and $f: R \longrightarrow R/s(R)$ be a natural ring homomorphism. Then

(i) if r is idempotent then a module P is (r,2,s,1)-projective if and only if P/ch(s)(P) is (f[r],2)-projective in R/s(R)-mod,

(ii) if r is a radical then a module P is (r,3,s,1)-projective if and only if P/ch(s)(P) is (f[r],3)-projective in R/s(R)-mod.

<u>Proof</u>: (i). Suppose P is (r,2,s,1)-projective and $0 \longrightarrow K \hookrightarrow Q \xrightarrow{g} P/ch(s)(P) \longrightarrow 0$ is a projective presentation of P/ch(s)(P) in R/s(R)-mod. Then $0 \longrightarrow K/\widetilde{f[r]}(K) \longrightarrow Q/\widetilde{f[r]}(K) \xrightarrow{\bar g} P/ch(s)(P) \longrightarrow 0$ ($\bar g$ induced by g) is a (f[r],2)-projective presentation in R/s(R)-mod by Proposition 1.1(ii). Consider the following diagram in R-mod

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & K/\tilde r(K) & \hookrightarrow & Q/\tilde r(K) & \xrightarrow{\bar g} & P/ch(s)(P) & \longrightarrow & 0 \end{array} \quad (\pi \text{ natural})$$

As it is easy to see $Q/\tilde r(K) \in \mathcal{F}_{ch(s)}$ and $K/\tilde r(K) \subseteq^{(r,2)} Q/\tilde r(K)$. Now P is (r,2,s,1)-projective and $\bar g \circ v = \pi$ for some $v \in \mathrm{Hom}_R(P, Q/\tilde r(K))$ which induces $\bar v: P/ch(s)(P) \longrightarrow Q/\tilde r(K)$ with $\bar g \circ \bar v = 1$. Thus $\bar g$ splits in R/s(R)-mod and consequently P/ch(s)(P) is (f[r],2)-projective in R/s(R)-mod.

Conversely, if

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow h & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

is a diagram in R-mod with exact row, Ker $g \subseteq^{(r,2)} M$, $M \in \mathcal{F}_{ch(s)}$ and if P/ch(s)(P) is (f[r],2)-projective in R/s(R)-mod, then

$$\begin{array}{ccccc} & & P/ch(s)(P) & & \\ & & \downarrow \bar{h} & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

($\bar{h}$ induced by h) is a diagram in R/s(R)-mod with Ker $g \subseteq^{(f[r],2)} M$, and hence $g \circ v = \bar{h}$ for some homomorphism $v: P/ch(s)(P) \longrightarrow M$. Thus $g \circ (v \circ \pi) = h$ ($\pi : P \longrightarrow P/ch(s)(P)$ is a natural homomorphism) and consequently P is (r,2,s,1)-projective.

(ii) Similarly as in (i).

Corollary 1.3. Let s be a preradical. Then a module P is (1,s)-projective if and only if P/ch(s)(P) is projective in R/s(R)-mod.

Proposition 1.4. Let r be a preradical for R-mod and $P \in$ R-mod. Then

(i) if r is idempotent then P is $(\tilde{r},1)$-projective if and only if it is (r,2)-projective,

(ii) if r is idempotent and $\tilde{r}$ is pseudohereditary then P is (r,2)-projective if and only if it is $(1,\tilde{r})$-projective,

(iii) if r is a radical then P is (r,3)-projective if and only if it is (1,r)-projective,

(iv) P is (3,r)-projective if and only if it is (2,r)-projective if and only if it is $(1,\tilde{r})$-projective.

Proof: (i). It suffices to prove the "only if part". Let P be (r,2)-projective and $0 \longrightarrow K \hookrightarrow Q \xrightarrow{g} P \longrightarrow 0$ be a projective presentation of P. Then $0 \longrightarrow K/\tilde{r}(K) \longrightarrow Q/\tilde{r}(K) \xrightarrow{\bar{g}} P \longrightarrow 0$ ($\bar{g}$ induced by g) is a $(\tilde{r},1)$-projective presentation of P with $K/\tilde{r}(K) \in \mathcal{F}_r$ by Proposition 1.1(i). Thus $\bar{g}$ splits and consequently P is $(\tilde{r},1)$-projective.

(ii) See Rangaswamy [14] Theorem 8 and Corollary 1.3.

(iii) With respect to Corollary 1.3 it suffices to prove that P is (r,3)-projective if and only if P/ch(r)(P) is projective in R/r(R)-mod. Let P be (r,3)-projective, $f: R \longrightarrow R/r(R) = \bar{R}$ be a natural ring homomorphism and $0 \longrightarrow K \longrightarrow Q \xrightarrow{g} P/ch(r)(P) \longrightarrow 0$ be a projective presentation in $\bar{R}$-mod. Then $Q \in \mathcal{F}_r$ since $f[r](Q) = f[r](\bar{R})\, Q$, and hence $g \circ v = \pi$ ($\pi : P \longrightarrow P/ch(r)(P)$ natural) for some $v \in Hom_R(P,Q)$ by the (r,3)-projectivity of P. Thus v induces $\bar{v}: P/ch(r)(P) \longrightarrow Q$ with $g \circ \bar{v} = 1$, hence g splits in R/r(R)-mod and consequently P/ch(r)(P) is projective in R/r(R)-mod.

We shall prove the sufficiency by modifying of the proof of Theorem 8 in [14]. Let P/ch(r)(P) be projective in R/r(R)-mod and $0 \longrightarrow K \hookrightarrow Q \xrightarrow{g} P \longrightarrow 0$ be a projective presentation of P. Then by Proposition 1.1 (iii) $0 \longrightarrow K/(r(Q) \cap K) \longrightarrow Q/(r(Q) \cap K) \xrightarrow{\bar{g}} P \longrightarrow 0$ is a (r,3)-projective presentation of P with $K' = K/(r(Q) \cap K) \subseteq^{(r,3)} Q/(r(Q) \cap K) = Q'$ ($\bar{g}$ induced by g).

Consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \longrightarrow & Q' & \xrightarrow{\bar{g}} & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \\
0 & \longrightarrow & (K' + ch(r)(Q'))/ch(r)(Q') & \longrightarrow & Q'/ch(r)(Q') & \xrightarrow{\bar{g}'} & P/ch(r)(P) & \longrightarrow & 0
\end{array}$$

where π_1, π_2 are natural epimorphisms.

As it is easy to see the right hand square is a pullback. Now $\bar{g}'$ splits since P/ch(r)(P) is projective in R/r(R)-mod, and hence $\bar{g}$ splits. Thus P is (r,3)-projective.

(iv) With respect to Proposition 2.9 in [11] it suffices to prove that P is (2,r)-projective implies P is (1,r)-projective for a radical r. It can be proved similarly as the necessity in (iii).

Corollary 1.5. Let r,s be preradicals for R-mod and P $\in$ $\in$ R-mod. Then

(i) if r is idempotent and every submodule of $\tilde{r}(R/s(R))$ is $\tilde{r}$-torsion then P is (r,2,s,1)-projective iff it is $(1,s\Delta\tilde{r})$-projective,

(ii) if r is a radical then P is (r,3,s,1)-projective iff it is $(1,s\Delta r)$-projective. •

Proposition 1.6. Let r,s be preradicals. Then every submodule of $\tilde{r}(R/s(R))$ is $\tilde{r}$-torsion, provided at least one of the following conditions is satisfied:

(i) r is hereditary,

(ii) s is idempotent and $s\Delta\tilde{r}$ is pseudohereditary.

Proof: Obvious.

§ 2. (r,i,s,j,M)-projective and strongly (r,i,s,j,M)-projective modules

Definition 2.1. Let r,s be preradicals, i,j $\in$ {1,2,3} and M $\in$ R-mod. A module P is said to be strongly (r,i,s,j,M)-projective if it is (r,i,s,j,M^I)-projective for every index set I.

If r = s = zer, then we obtain the strongly M-projectivity in the sense of K. Varadarajan (see [18]).

Let r,s be preradicals, $i,j \in \{1,2,3\}$. For any $P \in$ R-mod let us denote $C^p_{(r,i,s,j)}(P) = \{M \in$ R-mod, P is (r,i,s,j,M)-projective$\}$. Further the class of all (r,i,s,j,M)-projective modules will be denoted by $C_p^{(r,i,s,j)}(M)$.

Due to G. Azumaya an epimorphism $f:A \longrightarrow B$ is called an M-epimorphism if there exists $h:A \longrightarrow M$ with $\mathrm{Ker}\, f \cap \mathrm{Ker}\, h = 0$. These two following propositions are motivated by the results of G. Azumaya (see [18] Propositions 1.3 and 1.5). We include them here without the proof.

Proposition 2.2. Let r be a preradical and s be a cohereditary radical. Then the following are equivalent for a module P:

(i) P is (r,2,s,2,M)-projective,

(ii) given any M-epimorphism $f:A \longrightarrow B$ and any homomorphism $g:P \longrightarrow B$ with $r(\mathrm{Ker}\, f) = 0$ and $s(f^{-1}(\mathrm{Im}\, g)) = 0$, there exists a homomorphism $v:P \longrightarrow A$ such that $f \circ v = g$.

Proposition 2.3. Let r,s be preradicals and $P,M \in$ R-mod. Then

(i) $C_p^{(r,i,s,j)}(M)$ is closed under arbitrary direct sums and direct summands $i,j \in \{1,2,3\}$,

(ii) $C^p_{(r,2,s,2)}(P)$ is closed under submodules,

(iii) if r,s are idempotent $K \in \mathcal{F}_r \cap \mathcal{F}_s$ and $M \in C^p_{(r,2,s,2)}(P)$ then $M/K \in C^p_{(r,2,s,2)}(P)$,

(iv) if r,s are both cohereditary then $C^p_{(r,2,s,2)}(P)$ is closed under the formation of finite direct sums. Moreover, if P has a projective cover then $C^p_{(r,2,s,2)}(P)$ is closed under the formation of arbitrary direct products.

Proposition 2.4. Let r,s be preradicals. Then a module P is strongly (r,2,s,2,M)-projective if and only if it is $(r,2,s \triangle p^{\{M\}},2)$-projective.

Proof: Obvious.

Corollary 2.5. Let $M \in$ R-mod. Then the following are equivalent for a module P:

(i) P is strongly M-projective,

(ii) P is $(1,p^{\{M\}})$-projective,

(iii) P is $(2,p^{\{M\}})$-projective,

(iv) P is $(p^{\{M\}},3)$-projective,

(v) P is $(3,p^{\{M\}})$-projective,

(vi) P/(0:M) P is projective in R/(0:M)-mod.

Moreover, if M is pseudo-injective then the above stated conditions are equivalent to:

(vii) P is $(p^{\{M\}},2)$-projective,

(viii) P is $(p^{\{M\}},1)$-projective.

Proof: By Proposition 1.4 and Corollary 1.3.

Corollary 2.6. Let r be a preradical. Then there is a ch(r)-torsionfree module M such that a module P is (1,r)-projective if and only if it is strongly M-projective.

Proof: By [11] Proposition 2.9 (iv) P is (1,r)-projective iff it is (1,ch(r))-projective. Now by [2] Proposition 4.6 $ch(r) = p^{\{M\}}$, where $M = \prod_{A \in \mathcal{A}} A$, $\mathcal{A}$ is a representative set of ch(r)-torsionfree cocyclic modules and Corollary 2.5 finishes the proof.

Theorem 2.7. Let r be an idempotent preradical such that $\tilde{r}$ is pseudohereditary. Then there is a hereditary radical s such that a module P is (r,2)-projective if and only if it is (s,2)-projective.

Proof: By Proposition 1.4 (ii) and [11] Proposition 2.9 P is (r,2)-projective iff it is $(1, ch(\tilde{r}))$-projective. Now by [12] Proposition 1.5 $ch(\tilde{r}) = ch(p^{\{Q\}})$ where $Q = \prod_{A \in \mathcal{A}} E(A)$, $\mathcal{A}$ is a representative set of cyclic r-torsion-free modules. It is enough to put $s = p^{\{Q\}}$ and use [11] Proposition 2.9 (iv) and Corollary 2.5 (vii).

Proposition 2.8. Let r,s be preradicals. If M is a cogenerator for R-mod then a module P is strongly (r,2,s,2,M)-projective if and only if it is (r,2,s,2)-projective.

Proof: By Proposition 2.4.

M.S. Shrikhande calls a module cohereditary if every its factormodule is injective (see [15]).

Proposition 2.9. Let M be an injective module. Consider the following conditions:

(i) Every submodule of a strongly M-projective module is strongly M-projective.

(ii) Every submodule of a projective module is strongly M-projective.

(iii) M^I is cohereditary for every index set I.

(iv) R/(O:M) is a left hereditary ring.

Then conditions (i),(ii) and (iii) are equivalent and imply (iv).

Moreover, if $ch(p^{\{M\}})$ is hereditary then (iv) implies (i).

Proof: (i) is equivalent to (ii) and (ii) is equivalent to (iii). It immediately follows from [15] Theorem 3.2'.

(i) implies (iv). By Corollary 2.5 (vi).

(iv) implies (i). Use Corollary 2.5 (vi) and the fact that $ch(p^{\{M\}})$ is hereditary.

Corollary 2.10. R is a left hereditary ring if and only if $E(R)^I$ is cohereditary for every index set I.

The next Proposition is a modification of the well-known Theorem on test modules for projectivity (see [4] Theorem 10). We include it here without the proof for the sake of completeness.

Proposition 2.11. Let $M \in R$-mod. Then the following are equivalent:

(i) every strongly M-projective module is projective,

(ii) $(0:M) = p^{\{M\}}(R)$ is a ring direct summand of R and it is completely reducible ring.

Proposition 2.12. Let r be an idempotent cohereditary radical, s be a preradical and let P be a module possessing an (r,2)-projective cover $0 \longrightarrow K \longrightarrow Q \xrightarrow{\varphi} P \longrightarrow 0$. Then

(i) P is (r,2,s,1)-projective if and only if $Ker\,\varphi \subseteq ch(s)(Q)$,

(ii) P is (r,2,s,2)-projective if and only if $Ker\,\varphi \subseteq \tilde{s}(Q)$.

Proof: (i). By Proposition 1.1 (ii) $r(K) = 0$. Let P be (r,2,s,1)-projective. Consider the following commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & P \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ Q/ch(s)(Q) & \xrightarrow{\bar{\varphi}} & P/ch(s)(P) \end{array}$$

where π_1, π_2 are natural epimorphisms. Then $\bar{\varphi} \circ v = \pi_2$ for some $v: P \longrightarrow Q/ch(s)(Q)$ since $Ker\,\bar{\varphi} \in \mathcal{F}_r$, $Q/ch(s)(Q) \in \mathcal{F}_{ch(s)}$ and P is (r,2,s,1)-projective. Now, $\pi_1 = v \circ \varphi$ since $Ker\,\varphi + Ker\,(\pi_1 - v \circ \varphi) = Q$ as is easily seen. Therefore $Ker\,\varphi \subseteq ch(s)(Q)$.

The converse implication is obvious.

(ii) Similarly as in (i).

<u>Proposition 2.13</u>. Let r be an idempotent cohereditary radical, s be a preradical and let P be a module possessing an (r,2)-projective cover $0 \longrightarrow K \longrightarrow Q \xrightarrow{\varphi} P \longrightarrow 0$. Then

(i) $(Q/(ch(s)(Q) \cap \operatorname{Ker}\varphi), \bar{\varphi})$ where $\bar{\varphi}$ is induced by φ is an (r,2,s,1)-projective cover of P,

(ii) $(Q/(\tilde{s}(Q) \cap \operatorname{Ker}\varphi), \bar{\varphi})$ where $\bar{\varphi}$ is induced by φ is an (r,2,s,2)-projective cover of P.

<u>Proof</u>: Use Proposition 2.12.

<u>Proposition 2.14</u>. Let r be an idempotent cohereditary radical and s be a cohereditary radical. If a module P possesses an (r,2)-projective cover $0 \longrightarrow K \longrightarrow Q \xrightarrow{\varphi} P \longrightarrow 0$ then P is (r,2,s,2,M)-projective if and only if it is strongly (r,2,s,2,M)-projective.

<u>Proof</u>: Let P be (r,2,s,2,M)-projective. With respect to Propositions 2.4 and 2.12 it suffices to prove $\operatorname{Ker}\varphi \subseteq s \vartriangle p^{\{M\}}(Q)$. If $f: Q/s(Q) \longrightarrow M$ is arbitrary and

$$\begin{array}{ccc} Q/s(Q) & \xrightarrow{\bar{\varphi}} & P/s(P) \\ \downarrow f & & \downarrow g \\ M & \xrightarrow{h} & N \end{array}$$

is a push-out diagram ($\bar{\varphi}$ induced by φ), then $\operatorname{Ker} h \in \mathcal{F}_r$ and $h^{-1}(\operatorname{Im} g) \in \mathcal{F}_s$. Now consider the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & P \\ \downarrow f\circ\pi_1 & & \downarrow g\circ\pi_2 \\ M & \xrightarrow{h} & N \end{array}$$

where π_1, π_2 are natural epimorphisms. In the same way as in the proof of Proposition 2.12 we obtain $\mathrm{Ker}\,\varphi \subseteq \mathrm{Ker}\, f \circ \pi_1$, and hence $\mathrm{Ker}\,\varphi \subseteq s \Delta p^{\{M\}}(Q)$.

Corollary 2.15. Let r be an idempotent cohereditary radical, s be a cohereditary radical, $M \in$ R-mod and P be a module possessing an (r,2)-projective cover $0 \longrightarrow K \longrightarrow Q \xrightarrow{\varphi} P \longrightarrow 0$. Then $(Q/(s \Delta p^{\{M\}}(Q) \cap \mathrm{Ker}\,\varphi), \bar{\varphi})$ where $\bar{\varphi}$ is induced by φ is an (r,2,s,2,M)-projective cover of P.

Proof: By Propositions 2.13, 2.14 and 2.4.

§ 3. (r,i,s,j)-quasiprojective modules

Proposition 3.1. Let r,s be two cohereditary radicals and $Q_i \in$ R-mod $i \in \{1,2,\dots,n\}$. Then $Q_1 \oplus Q_2 \oplus \dots \oplus Q_n$ is (r,2,s,2)-quasi-projective if and only if Q_i is (r,2,s,2)-quasiprojective and $(r,2,s,2,Q_j)$-projective for every $i,j \in \{1,2,\dots,n\}$, $i \neq j$.

Proof: It follows immediately from Proposition 2.3 (i), (iv).

Proposition 3.2. Let r,s be two idempotent preradicals and Q be an (r,2,s,2)-quasiprojective module. If K is a characteristic submodule of Q such that $K \in \mathcal{F}_r \cap \mathcal{F}_s$ then Q/K is (r,2,s,2)-quasiprojective.

Proof: Obvious.

Proposition 3.3. Let r be an idempotent cohereditary radical and s be a cohereditary radical. If a module P possesses an (r,2)-projective cover $0 \longrightarrow K \longrightarrow Q \xrightarrow{\varphi} P \longrightarrow 0$ then $(Q/(s \Delta p^{\{P\}}(Q) \cap \mathrm{Ker}\,\varphi), \bar{\varphi})$ where $\bar{\varphi}$ is induced by φ is an (r,2,s,2)-quasiprojective cover of P.

Proof: Use Propositions 2.4, 2.13 and 2.14.

Corollary 3.4. Let r be an idempotent cohereditary radical, s be a cohereditary radical and $P \in$ R-mod possessing a projective cover $0 \longrightarrow K \longrightarrow Q \xrightarrow{\varphi} P \longrightarrow 0$. Then $(Q/(C_{s \wedge p^{\{P\}}}(r(\operatorname{Ker}\varphi) : Q) \cap \operatorname{Ker}\varphi), \overline{\varphi})$ where $\overline{\varphi}$ is induced by φ is an (r,2,s,2)-quasi projective cover of P.

Proof: By Proposition 3.3 and [11] Proposition 2.10 (vii).

Following closely the ideas of J.S. Golan (see [8]) we obtain Propositions 3.5 - 3.8 which are included here without the proof.

Proposition 3.5. Let r be an idempotent cohereditary radical. Then the following are equivalent:

(i) Every (finitely generated) R-module has an (r,2)-projective cover.

(ii) Every (finitely generated) R-module P has an (r,2)-quasiprojective cover $0 \longrightarrow K \longrightarrow Q \longrightarrow P \longrightarrow 0$ with $K \in \mathcal{F}_r$.

Proposition 3.6. Let r be a cohereditary splitting radical (i.e. every module splits in r). Then the following are equivalent:

(i) Every finitely presented R-module has an (r,2)-projective cover.

(ii) Every finitely presented R-module P has an (r,2)-quasiprojective cover $0 \longrightarrow K \longrightarrow Q \longrightarrow P \longrightarrow 0$ with $K \in \mathcal{F}_r$.

Proposition 3.7. Let r be an idempotent preradical for R-mod. Then $\overline{R} = R/\tilde{r}(R)$ is a completely reducible ring if and only if for every simple $\overline{R}$-module P $\overline{R} \oplus P$ is (2,r)-quasiprojective in R-mod.

Proposition 3.8. Let r be an idempotent preradical such that $\tilde{r}$ is pseudohereditary. Then the following are equivalent:

(i) Every R-module is (r,2)-projective,

(ii) every R-module is (r,2)-quasiprojective,

(iii) every finitely generated R-module is (r,2)-quasiprojective.

(iv) The class of all (r,2)-quasiprojective R-modules is closed under the formation of finite direct sums.

(v) $R/\bar{r}(R)$ is a completely reducible ring.

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