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CONCERNING INTERIOR MAPPING THEOREM

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Abstract: Let X, Y be real normed linear spaces with scalar product and $F:B(x_0,r) \longrightarrow Y$ be a Lipschitzian mapping which can be approximated by a family of linear, continuous, "uniformly" open mappings with a certain accuracy. Then it is proved that Fx_0 lies in int $\overline{R(F)}$, see Theorem 1. Furthermore, additional conditions satisfying $Fx_0 \in$ int R(F) are discussed. The proof of the quoted result is carried out by developing of the method of Pourciau [5, Section 9], where the finitely dimensional case is considered.

Key words: Space with scalar product, Lipschitzian mapping, convex closed set, interior(of the closure) of range, interior mapping theorem.

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Introduction. The well known interior mapping theorem, due to Graves [3, Theorem 1], asserts that $Fx_0 \in$ int R(F) if the mapping $F:X \longrightarrow Y$ does not differ much from a linear, continuous, open mapping L near x_0 and X is complete. Recently Pourciau [5] obtained the same conclusion provided that X and Y are finitely dimensional and the only mapping L is replaced by a family of linear, surjective (i.e., open) mappings. His result reads as follows:

Theorem (Pourciau [5, Theorem 6.1]). Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $m \leq n$, be a Lipschitzian mapping, $x_0 \in \operatorname{int} D(F)$, and let the Clarke subdifferential (cf. [1, Definition 1], [4, Section 2])

$$\partial F(x_0) = \operatorname{co} \{ \lim_{k \to \infty} dF(x_k) \mid x_k \longrightarrow x_0,\ dF(x_k) \text{ exist} \}$$

be surjective, i.e., each $L \in \partial F(x_0)$ is surjective.

Then $Fx_0 \in \operatorname{int} R(F)$.

It should be noted that, in the case $m = n$, the above result is contained in Clarke's inverse function theorem [1, Theorem 1].

The aim of this note is to extend, as long as we are able, the Pourciau theorem to infinitely dimensional spaces, see Theorem 1. In the proof we follow [5, Section 9], where a penalty functional technique is used. But some difficulties are to be avoided in our situation. Namely, in [5, Section 9], the Clarke subdifferential of some nonnegative continuous functional at a point of its minimum is computed with help of the chain rule [5, Proposition 4.8]. However, in our case no kind of differentiability is assumed and hence no chain rule is available. Moreover, in an infinitely dimensional space, it may happen that a functional on a closed ball attains minimum in no point.

The obtained result is, unfortunately, somewhat weaker than what we would wish. That is we get that $Fx_0 \in \operatorname{int} \overline{R(F)}$ only. In the last section there are given some additional conditions under which our result becomes an interior mapping theorem, i.e., $Fx_0 \in \operatorname{int} R(F)$.

Also the sense of the condition (2) is explained in this section.

Result. Let X, Y be real normed linear spaces with scalar products $\langle .,. \rangle$ and corresponding norms $\|\cdot\|$, i.e., $\langle u,u \rangle = \|u\|^2$. $B(x_0,r)$ stands for the open ball of centre $x_0 \in X$ and radius $r > 0$. The closure of a set $M \subset X$ is denoted by $\overline{M}$, the closed convex hull of M by $\overline{co}$ M. $M^{\perp}$ stands for the set

$$\{ x \in X \mid \langle x,m \rangle = 0 \text{ for all } m \in M \} .$$

Given a mapping $F:X \longrightarrow Y$, its domain and range are denoted by D(F) and R(F) respectively. The space of all continuous, linear mappings $L:X \longrightarrow Y$, with D(L) = X, endowed with the usual linear structure and norm is denoted by $\mathcal{L}(X,Y)$. The norm $\|L\|$ of $L \in \mathcal{L}(X,Y)$ is defined by

$$\|L\| = \sup \{ \|Lx\| \mid \|x\| = 1 \} .$$

L^* means the adjoint mapping to L, N(L) is the space of all $x \in X$ satisfying $Lx = 0$. $\mathbb{R}^n$ stands for the n-dimensional Euclidean space.

Theorem 1. Let X, Y be real normed linear spaces with scalar products and $F:X \longrightarrow Y$ be a mapping with $\overline{B(x_0,r)} \subset D(F)$ for some $x_0 \in X$ and some $r > 0$. Assume that there are numbers $\alpha > 0$, $\beta \in [0,\frac{\alpha}{2})$, $\gamma > 0$, and a set $\mathfrak{M} \subset \mathcal{L}(X,Y)$ such that the following three conditions are satisfied:

(1) $\qquad \forall x, \bar{x} \in \overline{B(x_0,r)} \quad \|F\bar{x} - Fx\| \leqq \gamma \|\bar{x} - x\|,$

(2) $\quad \forall y \in Y \quad \exists\, 0 \neq x \in X \quad \forall L \in \mathfrak{M} \ \langle y, Lx \rangle \geqq \alpha \|y\| \|x\|,$

(3) $\quad \forall x, \bar{x} \in \overline{B(x_0,r)} \quad \exists L \in \mathfrak{M} \quad \|F\bar{x} - Fx - L(\bar{x} - x)\| \leqq \beta \|\bar{x} - x\|.$

Then $Fx_0 \in \operatorname{int} \overline{R(F)}$; more precisely,

$$B(Fx_0, (\frac{\alpha}{2} - \beta)r) \subset \overline{F(\overline{B(x_0,r)})}.$$

Proof: Fix $y \in B(Fx_0, (\frac{\alpha}{2} - \beta)r)$, $y \neq Fx_0$, arbitrarily. We shall argue by contradiction, that is, let there be $\rho > 0$ such that

(4) $$\forall x \in \overline{B(x_0,r)} \quad \| Fx - y \| \geq \rho > 0.$$

We shall consider the following functional

$$\varphi(x) = \| Fx - y \| + k \| x - x_0 \| , \quad x \in \overline{B(x_0,r)},$$

where

(5) $$k = \frac{2}{r} \| Fx_0 - y \| .$$

(We note that the member $k \| x - x_0 \|$ plays the role of a "penalty".) Denote

$$m = \inf \{ \varphi(x) \mid x \in \overline{B(x_0,r)} \} .$$

At this point the proof splits into two cases. First let us assume that

(6) $$m < \| Fx_0 - y \| \; [= \varphi(x_0)] .$$

We remark that $k < \alpha - 2\beta$ for $\| Fx_0 - y \| < (\frac{\alpha}{2} - \beta)r$. Choose

(7) $$\Delta \in (0, \alpha - 2\beta - k) \cap (0, \tfrac{1}{2}(\| Fx_0 - y \| - m))$$

and denote

$$M = \{ x \in \overline{B(x_0,r)} \mid \varphi(x) < m + \Delta \} .$$

We claim

(8) $$M \subset \{ x \in B(x_0, \tfrac{r}{2}) \mid \| x - x_0 \| > \frac{\Delta}{\gamma - k} \} .$$

(In the sequel we shall show that $\gamma - k > 0$.) Indeed, let $x \in M$. If $\| x - x_0 \| \geq \frac{r}{2}$, it would then follow by (4),(5) and (7) that

$$\Delta + m > \varphi(x) = \| Fx - y \| + k \| x - x_0 \| > k \| x - x_0 \| \geq k\frac{r}{2} =$$

$$= \| Fx_0 - y \| > 2\Delta + m,$$

a contradiction. Hence $x \in B(x_o, \frac{r}{2})$. Also, (1) and (7) yield

$$\Delta + m > \| Fx - y \| + k \| x - x_o \| \geqq \| Fx_o - y \| - \| Fx - Fx_o \| + $$
$$+ k \| x - x_o \| \geqq \| Fx_o - y \| - (\gamma - k) \| x - x_o \| > 2\Delta + m -$$
$$- (\gamma - k) \| x - x_o \| , \quad (\gamma - k) \| x - x_o \| > \Delta ,$$

which completes the proof of (8). The last inequality also shows that $\gamma > k$.

Fix $x \in M$ and $h \in B(0, \frac{r}{2})$. By (8), $x + h \in B(x_o, r)$. We shall approximate the difference $\varphi(x + h) - \varphi(x)$ with help of some linear mapping. For brevity put

$$a = Fx - y, \quad b = F(x + h) - y$$

and choose some $L \in \mathfrak{M}$ which corresponds to x, $x + h$ by (3). Then (1),(3) and (4) yield

$$\| b \| - \| a \| - \frac{\langle Lh, a \rangle}{\| a \|} = \frac{1}{\| a \| + \| b \|} (\| b - a \|^2 + 2 \langle b - a - Lh, a \rangle) +$$

$$+ \langle Lh, a \rangle \frac{\| a \| - \| b \|}{\| a \| (\| a \| + \| b \|)} \leqq \frac{1}{\| a \| + \| b \|} (\gamma^2 \| h \|^2 + 2\beta \| h \| \| a \|) +$$

$$+ \| L \| \| h \| \frac{\gamma \| h \|}{\| a \| + \| b \|} \leqq \frac{\gamma}{2\varrho} (\gamma + \| L \|) \| h \|^2 + 2\beta \| h \| ,$$

i.e.,

$$\| F(x + h) - y \| - \| Fx - y \| - \frac{\langle Lh, Fx - y \rangle}{\| Fx - y \|} \leqq \tag{9}$$
$$\leqq \frac{\gamma}{2\varrho} (\gamma + \| L \|) \| h \|^2 + 2\beta \| h \| .$$

Similarly, as $\| x - x_o \| > \frac{\Delta}{\gamma - k}$ owing to (8), we have

$$\| x + h - x_o \| - \| x - x_o \| - \frac{\langle h, x - x_o \rangle}{\| x - x_o \|} = \frac{\| h \|^2}{\| x + h - x_o \| + \| x - x_o \|} +$$

$$+ \frac{\langle h, x - x_o \rangle}{\| x - x_o \|} \cdot \frac{\| x - x_o \| - \| x + h - x_o \|}{\| x + h - x_o \| + \| x - x_o \|} \leqq \frac{2 \| h \|^2}{\| x - x_o \|} \leqq 2 \frac{\gamma - k}{\Delta} \| h \|^2 .$$

Thus, adding the last two inequalities, we get

$$\varphi(x + h) - \varphi(x) - \frac{\langle Lh, Fx - y\rangle}{\| Fx - y\|} - k\frac{\langle h, x - x_0\rangle}{\| x - x_0\|} \leqq$$

$$\leqq \left[\left(\frac{\gamma^2}{2\varrho} + \frac{\gamma\| L\|}{2\varrho} + 2\,\frac{\gamma - k}{\Delta} \right) \| h\| + 2\beta \right] \| h\| .$$

Furthermore there is (see (7)) $\delta \in (0, \frac{r}{2})$ such that

$$\left(\frac{\gamma^2}{2\varrho} + \frac{\gamma\| L\|}{2\varrho} + 2\,\frac{\gamma - k}{\Delta} \right) \delta < \alpha - \Delta - 2\beta - k.$$

Thus we get that

(10) $\quad \varphi(x + h) - \varphi(x) - \frac{\langle Lh, Fx - y\rangle}{\| Fx - y\|} \leqq (\alpha - \Delta)\| h\|$

whenever $x \in M$, $h \in B(0, \delta)$ and L corresponds to x, $x + h$ by (3).

Now let $\bar{x} \in M$ be such that $\varphi(\bar{x}) < m + \frac{1}{2}\delta\Delta$. By (2), there is $\bar{h} \in X$, $\|\bar{h}\| = \frac{3}{4}\delta$, such that

(11) $\quad \langle y - F\bar{x}, L\bar{h} \rangle \geqq \alpha \| y - F\bar{x}\| \, \|\bar{h}\| = \frac{3}{4}\alpha\delta \| y - F\bar{x}\|$

for all $L \in \mathfrak{M}$. Let $\bar{L} \in \mathfrak{M}$ correspond to $\bar{x}$, $\bar{x} + \bar{h}$ by (3). Then, bearing in mind that

(12) $\quad \varphi(\bar{x} + \bar{h}) - \varphi(\bar{x}) > m - (m + \frac{1}{2}\delta\Delta) = -\frac{1}{2}\delta\Delta ,$

we get from (10) - (12) that

$$-\frac{1}{2}\delta\Delta + \frac{3}{4}\alpha\delta = -\frac{1}{2}\delta\Delta + \alpha\|\bar{h}\| < \varphi(\bar{x} + \bar{h}) - \varphi(\bar{x}) -$$

$$- \frac{\langle F\bar{x} - y, \bar{L}\bar{h}\rangle}{\| F\bar{x} - y\|} \leqq (\alpha - \Delta)\|\bar{h}\| = (\alpha - \Delta)\frac{3}{4}\delta ,$$

$$\frac{3}{4}\delta\Delta < \frac{1}{2}\delta\Delta ,$$

a contradiction.

It remains to investigate the second case, that is

$m = \|Fx_0 - y\|$. It is easy to check that (9) also holds for $x = x_0$, all $h \in B(x_0,r)$ and corresponding $L \in \mathfrak{M}$. Thus we get

$$\varphi(x_0 + h) - \varphi(x_0) - \frac{\langle Lh, Fx_0 - y\rangle}{\|Fx_0 - y\|} - k\|h\| \leqq \left[\frac{\gamma}{2\varrho} (\gamma + \|L\|) \|h\| + 2\beta \right] \|h\| .$$

Let $\delta_0 \in (0,r)$ be so small that

$$\frac{\gamma}{2\varrho}(\gamma + \|L\|)\, \delta_0 < \alpha - 2\beta - k.$$

Then, recalling that $\varphi(x_0 + h) \geqq \varphi(x_0)$, we get from the last two inequalities that

$$- \frac{\langle Lh, Fx_0 - y\rangle}{\|Fx_0 - y\|} < \alpha\|h\|$$

whenever $0 \neq h \in B(0, \delta_0)$ and L corresponds to x_0, $x_0 + h$ by (3). Following (2) there is $0 \neq h_0 \in B(0, \delta_0)$ such that

$$\langle y - Fx_0, Lh_0\rangle \geqq \alpha\|y - Fx_0\| \, \|h_0\|$$

for all $L \in \mathfrak{M}$. Combining the last two inequalities we get that $\alpha\|h_0\| < \alpha\|h_0\|$, a contradiction.

Thus, provided that (4) holds, we have obtained in both cases, that is $m < \|Fx_0 - y\|$ and $m = \|Fx_0 - y\|$, a contradiction. Whence it follows that

$$\inf\{\|Fx - y\| \mid x \in \overline{B(x_0,r)}\} = 0, \text{ i.e., } y \in \overline{F(B(x_0,r))}.$$

Q.E.D.

<u>Discussion</u>. The condition (2) looks somewhat curiously. Its sense is clarified in the following proposition. We show there that (2) means that the set $\overline{\text{co}}\, \mathfrak{M}$ consists of "uniformly" open mappings, or that the set of adjoint mappings

$$(\overline{co}\ \mathfrak{M})^* = \{L^* \mid L \in \overline{co}\ \mathfrak{M}\}$$

is "uniformly" injective. It should be noted that a condition similar to (2) can be found in Clarke [1, Lemma 3].

Proposition 1. Let X, Y be real Hilbert spaces, $\alpha > 0$ and $\mathfrak{M} \subset \mathcal{L}(X,Y)$. Then the following three assertions are equivalent each to other:

(i) $\forall y \in Y\ \exists 0 \neq x \in X\ \ \forall L \in \mathfrak{M}\ \ \langle y, Lx\rangle \geqq \alpha \|y\|\, \|x\|$

(ii) $\forall y \in Y\ \forall L \in \overline{co}\,\mathfrak{M}\ \ \|L^*y\| \geqq \alpha \|y\|$

(iii) $\forall y \in Y\ \ \forall L \in \overline{co}\ \mathfrak{M}\ \ \exists x \in X\ \ Lx = y \ \&\ \|y\| \geqq \alpha \|x\|$.

Proof: (i) $\Longrightarrow$ (ii). (i) obviously remains true if $\mathfrak{M}$ is replaced by $\overline{co}\ \mathfrak{M}$. That is, to each $y \in Y$ there is $0 \neq x \in X$ such that $\langle y, Lx\rangle \geqq \alpha \|y\|\, \|x\|$ whenever $L \in \overline{co}\ \mathfrak{M}$. Hence

$$\|L^*y\|\, \|x\| \geqq \langle L^*y, x\rangle = \langle y, Lx\rangle \geqq \alpha \|y\|\, \|x\|$$

and, dividing it by $\|x\| \neq 0$, (ii) follows.

(ii) $\Longrightarrow$ (i). The proof is similar to that of [1, Lemma 3]. Fix $y \in Y$. Since the case $y = 0$ is trivial, we may assume $y \neq 0$ in the sequel. The set

$$((\overline{co}\ \mathfrak{M})^*)y = \{L^*y \mid L \in \overline{co}\ \mathfrak{M}\}$$

is convex and, by (ii), is disjoint with $B(0, \alpha\|y\|)$. Hence, owing to the theorem on separation of two convex sets [6, 3.4 Theorem], there is $0 \neq x \in X$ such that

$$\alpha \|x\|\, \|y\| = \sup\{\langle x,v\rangle \mid v \in B(0, \alpha\|y\|)\} \leqq \inf\{\langle x,v\rangle \mid v \in ((\overline{co}\mathfrak{M})^*)y\}.$$

Whence it follows

$$\alpha \|x\|\, \|y\| \leqq \langle L^*y, x\rangle = \langle y, Lx\rangle$$

whenever $L \in \mathfrak{M}$ as (i) asserts.

(ii) $\Longrightarrow$ (iii). Fix $L \in \overline{co}\ \mathfrak{M}$. We remark that $\overline{R(L^*)}$ =

$= N(L)^{\perp}$ [6, 12.10 Theorem]. But (i) ensures that $R(L^*)$ is closed. Hence $X = R(L^*) \oplus N(L)$. Take $0 \neq x \in R(L^*)$ arbitrarily. Then $x = L^*y$ for some $y \in Y$ and so, by (ii),

$$\alpha \|x\|^2 = \alpha \langle L^*y, x\rangle = \alpha \langle y, Lx\rangle \leqq \alpha \|y\| \|Lx\| \leqq \|L^*y\| \|Lx\| =$$
$$= \|x\| \|Lx\|$$

and, cancelling it by $\|x\| \neq 0$, we get

(13) $$\forall x \in R(L^*) \quad \alpha \|x\| \leqq \|Lx\| .$$

It follows that L maps the closed subspace $R(L^*)$ of X onto a closed subspace of Y. On the other hand we always have

$$R(L) = L(X) = L(N(L)^{\perp}) = L(R(L^*)).$$

Hence $R(L)$ is closed in Y. Finally, as $\overline{R(L)} = N(L^*)^{\perp}$ [6, 12.10 Theorem] and $N(L^*) = \{0\}$ by (ii), we infer that $R(L) = Y$. Let now $y \in Y$ be given. There is $x \in R(L^*)$ such that $Lx = y$ and (13) completes the proof of (iii).

(iii) $\Longrightarrow$ (ii). Let $y \in Y$, $L \in \overline{co}\, \mathfrak{M}$. We may assume $y \neq 0$. By (iii), there is $0 \neq x \in X$ such that $Lx = y$ and $\|y\| \geqq \alpha \|x\|$. Hence

$$\|x\| \|L^*y\| \geqq \langle x, L^*y\rangle = \langle Lx, y\rangle = \|y\|^2 \geqq \alpha \|x\| \|y\|,$$
$$\|L^*y\| \geqq \alpha \|y\| .$$

Q.E.D.

If $F(\overline{B(x_0,r)})$ is closed, then our result becomes an interior mapping theorem. Let us formulate some additional conditions satisfying $F(\overline{B(x_0,r)})$ to be closed.

Proposition 2. $F(\overline{B(x_0,r)})$ is closed if one of the following conditions is fulfilled:

(i) X is complete (i.e., Hilbert) and there is $\delta > 0$ so that

(14) $\forall\, x,\ \bar{x} \in \overline{B(x_0,r)} \quad \|F\bar{x} - Fx\| \geqq \sigma \|\bar{x} - x\|$.

(ii) X is complete and each $L \in \mathcal{M}$ is injective (and hence an isomorphism thanks to Proposition 1)

(iii) $F = \lambda \mathrm{Id} + K$, where $\lambda \in \mathbb{R}$ and K is a compact mapping

(iv) $\dim X < +\infty$ (and hence $\dim Y \leqq \dim X$ owing to Proposition 1).

Proof: (i) is obvious. (ii). Let $x,\ \bar{x} \in \overline{B(x_0,r)}$ and take a corresponding L by (3). As L is injective, we have from Proposition 1 (iii) that

$\|F\bar{x} - Fx\| \geqq \|L(\bar{x} - x)\| - \|F\bar{x} - Fx - L(\bar{x} - x)\| \geqq (\alpha - \beta)\|\bar{x} - x\|$.

Now (i) can be used. (iii). The case $\lambda = 0$ is obvious. If $\lambda \neq 0$, see [2, III, 5 Proposition] for instance. (iv) follows from (iii) at once. Q.E.D.

It should be noted that, if (14) is satisfied for some $\sigma > 0$, then there exists a simpler proof of Theorem 1. Namely, we can use the functional $\varphi(x) = \|y - Fx\|^2$, which has no penalty member.

The case (iv) in the above proposition leads to the theorem of Pourciau. Let us show it. As the set $\partial F(x_0)$ is compact in the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, and surjective, there exists $\varepsilon > 0$ so that each L belonging to the set

$$\mathcal{M} = \{ L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \mid \exists \bar{L} \in \partial F(x_0) \quad \|L - \bar{L}\| \leqq \varepsilon \}$$

is still surjective. Since the multivalued mapping ∂F is upper semicontinuous [5, Proposition 4.1], there exists $r > 0$ such that $\partial F(x) \subset \mathcal{M}$ whenever $x \in \overline{B(x_0,r)}$. We note that $\mathcal{M}$ is closed and convex. Hence, by [5, Theorem 3.1, Proposition 3.2], to each $x,\ \bar{x} \in \overline{B(x_0,r)}$, there is $L \in \mathcal{M}$ so that

$$F\bar{x} - Fx = L(\bar{x} - x).$$

Thus (3) is satisfied with $\beta = 0$. (1) holds with some $\gamma > 0$ because F is a Lipschitzian mapping. Finally $\mathfrak{M}$ is convex compact since so is $\partial F(x_0)$, and each $L \in \mathfrak{M}$ is surjective, i.e., each L^* is injective. It follows there exists $\alpha > 0$ so that the assertion (ii) in Proposition 1 holds. Thus Proposition 1 yields (2), We have verified all the assumptions of Theorem 1 and so, together with Proposition 2 (iv), we get that Fx_0 lies in int R(F).

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