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PERIODIC SOLUTIONS TO THE INHOMOGENEOUS
SINE-GORDON EQUATION
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Abstract: For α and h satisfying certain conditions and for every ε sufficiently close to 0 it is shown that there exists a function $u \in C^2$
which fulfils
$u_{tt} - u_{xx} = \varepsilon (h(t,x) + \alpha \sin u)$, $u(t,0) = u(t,\pi) = 0$
and $u(t + 2\pi ,x) = u(t,x)$.

Key words: Weakly nonlinear wave equation, periodic solutions.

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In [1], O. Vejvoda derived sufficient conditions for

the existence of 2π-periodic solutions to the problem

(1) $u_{tt}(t,x) - u_{xx}(t,x) = \varepsilon f(t,x,u,\varepsilon)$, $t \in R$, $x \in \langle 0,\pi \rangle$

(2) $u(t,0) = u(t,\pi) = 0$, $t \in R$

and studied the problem with $f(t,x,u,\varepsilon) = h(t,x) + \alpha u +$
$+ \beta u^3$ in detail.

In this paper the same problem with $f(t,x,u, \varepsilon) =$
$= h(t,x) + \alpha \sin u$ is treated. In the sequel the functions
u and f are supposed to be extended in x on R by

$$u(t,x) = -u(t,-x) = u(t,x + 2\pi), f(t,x,u,\varepsilon) =$$
$$= -f(t,-x,-u, \varepsilon) = f(t,x + 2\pi ,u, \varepsilon).$$

The extended functions will be denoted again by u and f.

Let us note that if $u(t,x) = u(t, \pi - x)$ then $u(t,x + \pi) = -u(t,x)$. Put

$$C^{2*}_{2\pi}(R) = \{s \in C^2(R); \; s(x + \pi) = -s(x)\},$$

$$C^{2*}_e([0,2\pi] \times R) = \{u \in C^2([0,2\pi] \times R); \; u(t,x) = -u(t,-x) = u(t, \pi - x)\},$$

$$C^{2*}_{2\pi,e}(R^2) = \{u \in C^2(R^2); \; u(t,x) = u(t + 2\pi,x) = -u(t,-x) = u(t,x + 2\pi) = u(t,\pi - x)\}$$

and equip these spaces with the usual norms in which they are Banach spaces.

Let us first recall the result which is the starting point of our investigation.

<u>Theorem 1</u> (cf. Theorem 4.1.3 in [1]).

(i) Let a function f be continuous together with its derivatives

$$\frac{\partial^{j+k}f}{\partial x^j u^k} , \; j + k \leq 3, \; j \leq 2$$

on $R \times \langle 0,\pi \rangle \times R \times \langle -\varepsilon_0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$.

(ii) Let $f(t,0,0,\varepsilon) = f(t,\pi,0,\varepsilon) =$

$$\frac{\partial^2 f}{\partial x^j u^k}(t,0,0,\varepsilon) = \frac{\partial^2 f}{\partial x^j u^k}(t,\pi,0,\varepsilon) = 0,$$

$j + k = 2$.

(iii) Let $f(t,x,u,\varepsilon)$ be 2π-periodic in t.

(iv) Let the equation

$$(3) \quad G(s)(x) = \int_0^{2\pi} f(\pi,x - \tau, \; s(x) - s(2\tau - x),0)d\tau = 0$$

have a solution $s^* \in C^{2^*}_{2\pi}(R)$,

(v) Let there exist $[G_s'(s^*)]^{-1} \in L(C^{2^*}_{2\pi}(R),\ C^{2^*}_{2\pi}(R))$

where $C^{2^*}_{2\pi}(R) \supset \mathbb{R}(\Gamma)$, while

$$\Gamma(u,\varepsilon)(x) = \int_0^{2\pi} f(\tau, x-\tau, u(\tau, x-\tau), \varepsilon)d\tau \ ,$$

$$u \in C^{2^*}_e([0,2\pi] \times R),\ \varepsilon \in \langle -\varepsilon_0, \varepsilon_0 \rangle\ .$$

Then for sufficiently small ε the problem (1),(2) has a solution $u^* \in C^{2^*}_{2\pi,e}(R^2)$.

Our aim is to prove the following theorem:

Theorem 2.

(a) Let $h(t,x)$ together with its derivatives $\dfrac{\partial h}{\partial x}$, $\dfrac{\partial^2 h}{\partial x^2}$ be continuous on $R \times \langle 0, \pi \rangle$.

(b) Let $h(t,0) = h(t,\pi) = \dfrac{\partial^2 h}{\partial x^2}(t,0) = \dfrac{\partial^2 h}{\partial x^2}(t,\pi) = 0$ and $h(t, \pi - x) = h(t,x)$.

(c) Let h be 2π -periodic in t.

(d) If $H(x) = \displaystyle\int_0^{2\pi} h(\tau, x-\tau)d\tau \not\equiv 0$, then let

$$\alpha \geq 2 \cdot \|H\|_0^2 \cdot [\int_0^{2\pi} \sqrt{2 \cdot \|H\|_0^2 - H^2(x)}\ dx]^{-1},$$

where $\|H\|_0 = \max_{x \in R} |H(x)|$.

Then for sufficiently small ε the problem (1),(2) with $f(t,x,u,\varepsilon) = h(t,x) + \alpha \sin u$ has a solution $u \in C^{2^*}_{2\pi,e}(R^2)$.

Proof: The assumptions (i),(ii),(iii) of Theorem 1 are immediate consequences of the hypotheses (a),(b),(c) of the present theorem.

(iv) Denoting $I = \int_0^{2\pi} \cos s(\xi)\,d\xi$, we rearrange the equation (3) into the form

(4) $G(s)(x) = \alpha I \sin s(x) + H(x) = 0.$

For a while let us consider the functional I as a known constant. If $H(x) \equiv 0$, put $s^*(x) \equiv 0$. In the opposite case let us suppose that $|H(x)| < |\alpha I|$ and put $\hat{s}(I,x) = -\arcsin (\alpha I)^{-1} H(x)$. Clearly $\hat{s}$ is a solution to (4) if and only if

(5) $I = \int_0^{2\pi} \cos \hat{s}(I,\xi)\,d\xi = \int_0^{2\pi} (1-(\alpha I)^{-2} H^2(x))^{1/2}\,d\,x \equiv p(I).$

Evidently $p(2\pi) < 2\pi$ and (d) implies $p(\sqrt{2}\alpha^{-1} \| H\|_0) \geq \geq \sqrt{2}\alpha^{-1} \| H\|_0$. So (5) has at least one solution $I = I^*$ satisfying $|I^*| \geq \alpha^{-1} \| H\|_0$. Setting $s^*(x) = \hat{s}(I^*,x) \in \in C_{2\pi}^{2*}(R)$ we obtain a solution to (4) and the assumption (iv) is verified.

To prove (v) it suffices to show that for every $\varphi \in C_{2\pi}^{2*}(R)$, the equation

$G_s'(s^*)(\sigma)(x) \equiv \alpha I^* \sigma'(x) \cos s^*(x) - \alpha J \sin s^*(x) =$
$= \varphi(x) \in C_{2\pi}^{2*}(R),$

where $J = \int_0^{2\pi} \sin s^*(\xi) \cdot \sigma'(\xi)\,d\xi$, has a unique solution $\sigma(x) \in C_{2\pi}^{2*}(R)$ with $\| \sigma \|_2 \leq C \| \varphi \|_2$, C being a constant. We obtain easily that

$\sigma'(x) = (\varphi(x) + \alpha J^* \sin s^*(x))(\alpha I^* \cos s^*(x))^{-1} \in$
$\in C_{2\pi}^{2*}(R)$

with

$J^* = (\alpha I^* (1 - \int_0^{2\pi} \sin^2 s^*(\xi)(I^* \cos s^*(\xi))^{-1}\,d\xi))^{-1} \cdot$
$\cdot \int_0^{2\pi} \varphi(\xi) \cdot \operatorname{tg} s^*(\xi)\,d\xi =$

$$= (\propto I^{*2} \int_0^{2\pi} (1-2(\propto I^*)^{-2} H^2 (\xi))(1-(\propto I^*)^{-2} H^2 (\xi))^{-1/2} d\xi)^{-1}.$$

$$\cdot \int_0^{2\pi} \wp (\xi) tg \, s^* (\xi) \, d\xi$$

(by (d) this expression has sense).

Evidently $\| \sigma \|_2 \leqq C \| \wp \|_2$. This completes the proof.

R e f e r e n c e

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              nonlinear wave equation in one dimension, I.,
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