

Werk

Label: Article

Jahr: 1979

PURL: https://resolver.sub.uni-goettingen.de/purl?316342866_0020|log31

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PSEUDOHEREDITARY AND PSEUDOCOHEREDITARY PRERADICALS
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Abstract: M.L. Teply in [12] calls a torsion theory $(\mathcal{T}, \mathcal{F})$ pseudohereditary, if every submodule of $\mathcal{T}(R)$ is \mathcal{T} -torsion. In this paper, pseudohereditary preradicals together with the related dual problems are studied.

Key words: Preradical, pseudohereditary and pseudocohereditary preradicals, injective and projective modules.

AMS: Primary 18E40

Secondary 16A50, 16A52

Throughout this paper, R stands for an associative ring with unit element and $R\text{-mod}$ denotes the category of all unitary left R -modules. The injective hull of a module M will be denoted by $E(M)$, the direct product (sum) by $\prod_{i \in I} M_i$ ($\sum_{i \in I}^{\oplus} M_i$). A submodule N of M is called essential (superfluous) in M , if $K \cap N = 0$ implies $K = 0$ ($K + N = M$ implies $K = M$) for every submodule K of M . If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is a short exact sequence of R -modules, then we shall say that B is an envelope of A (B is a cover of C), if $f(A)$ is essential in B ($f(A)$ is superfluous in B). A ring is called left perfect, if every module has a projective cover.

We start with some basic definitions from the theory of

preradicals (for details see [1], [2] and [3]).

A preradical r for R -mod is a subfunctor of the identity functor, i.e. r assigns to each module M its submodule $r(M)$ in such a way that every homomorphism of M into N induces a homomorphism of $r(M)$ into $r(N)$ by restriction.

A preradical r is said to be

- idempotent if $r(r(M)) = r(M)$ for every module M ,
- a radical if $r(M/r(M)) = 0$ for every module M ,
- hereditary if $r(N) = N \cap r(M)$ for every submodule N of a module M ,
- cohereditary if $r(M/N) = (r(M) + N)/N$ for every submodule N of a module M ,
- faithful if $r(M) = 0$ for every projective module M ,
- cofaithful if $r(M) = M$ for every injective module M .

As it is easy to see a preradical r is faithful if and only if $r(R) = 0$ and r is cofaithful if and only if $r(E(R)) = E(R)$. A module M is r -torsion if $r(M) = M$ and r -torsionfree if $r(M) = 0$. We shall denote by \mathcal{T}_r (\mathcal{F}_r) the class of all r -torsion (r -torsionfree) modules. If r and s are preradicals then we write $r \leq s$ if $r(M) \subseteq s(M)$ for all $M \in R$ -mod. The idempotent core \bar{r} of a preradical r is defined by $\bar{r}(M) = \sum K$, where K runs through all r -torsion submodules K of M , and the radical closure \tilde{r} is defined by $\tilde{r}(M) = \bigcap L$, where L runs through all submodules L of M with M/L r -torsionfree. Further, the hereditary closure $h(r)$ is defined by $h(r)(M) = M \cap r(E(M))$ and the cohereditary core $ch(r)$ by $ch(r)(M) = r(R) \cap M$. The intersection (sum) of a family of preradicals $r_i, i \in I$ is a preradical defined by $(\bigcap_{i \in I} r_i)(M) = \bigcap_{i \in I} r_i(M)$ $((\sum_{i \in I} r_i)(M) = \sum_{i \in I} r_i(M))$. For a preradical r and modules

$N \subseteq M$ let us define $C_r(N:M)$ by $C_r(N:M)/N = r(M/N)$. For an arbitrary class of R -modules \mathcal{A} we define $p_{\mathcal{A}}(N) = \sum \text{Im } f$, f ranging over all $f \in \text{Hom}_R(M, N)$, $M \in \mathcal{A}$ and $p^{\mathcal{A}}(N) = \bigcap \text{Ker } f$, f ranging over all $f \in \text{Hom}_R(N, M)$, $M \in \mathcal{A}$. It is easy to see that $p_{\mathcal{A}}$ is an idempotent preradical ($p^{\mathcal{A}}$ is a radical). Moreover, if M is an injective (projective) module, then $p^{\{M\}}$ is hereditary ($p_{\{M\}}$ is cohereditary). Further, M is a faithful module if and only if $p^{\{M\}}$ is faithful. Dually, M is a cofaithful module if and only if $p_{\{M\}}$ is cofaithful.

§ 1. Pseudohereditary preradicals

Definition 1.1. A preradical r is said to be pseudohereditary if every submodule of $r(R)^{(I)}$ is r -torsion for every finite index set I .

Proposition 1.2. Let r be a preradical. Then the following are equivalent:

- (i) r is pseudohereditary,
- (ii) $N \subseteq \text{ch}(r)(M)$ implies $N \in \mathcal{T}_r$ for every submodule N of a module M ,
- (iii) $r(N) \subseteq \text{ch}(r)(M)$ implies $r(N) = N \cap \text{ch}(r)(M)$ for every submodule N of a module M .

Proof: (i) implies (ii). Let $M \in R\text{-mod}$ and $N \subseteq \text{ch}(r)(M)$. There is an epimorphism $f: F \rightarrow M$ with F free. Consider the epimorphism $\bar{f}: r(F) \rightarrow \text{ch}(r)(M)$ induced by f . By (i) $\bar{f}^{-1}(N) \in \mathcal{T}_r$ and hence $N = \bar{f}(\bar{f}^{-1}(N)) \in \mathcal{T}_r$.

(ii) implies (iii). If $M \in R\text{-mod}$, $N \subseteq M$ such that $r(N) \subseteq \text{ch}(r)(M)$ then $r(N) \subseteq \text{ch}(r)(M) \cap N$. By (ii) $K = \text{ch}(r)(M) \cap N \in \mathcal{T}_r$ and hence $\text{ch}(r)(M) \cap N \subseteq r(N)$.

(iii) implies (i). If $K \subseteq r(R)^{(I)}$ then clearly $r(K) \subseteq \text{ch}(r)(R)^{(I)}$

and (iii) yields $r(K) = \text{ch}(r)(R^{(I)}) \cap K = K$.

Proposition 1.3. Let r be a preradical. Then

- (i) if r is pseudohereditary, then $F \in \mathcal{F}_r$ implies $E(F) \in \mathcal{F}_{\text{ch}(r)}$,
- (ii) if r is a radical and $F \in \mathcal{F}_r$ implies $E(F) \in \mathcal{F}_{\text{ch}(r)}$ for every module F , then r is pseudohereditary.

Proof: (i). Let $F \in \mathcal{F}_r$. Since r is pseudohereditary, we have $K = F \cap \text{ch}(r)(E(F)) \in \mathcal{F}_r$. Thus $K = 0$ and $\text{ch}(r)(E(F)) = 0$.

(ii) Let $M \in R\text{-mod}$ and $N \subseteq \text{ch}(r)(M)$. Consider the following commutative diagram

$$\begin{array}{ccc} N/r(N) & \xrightarrow{g} & \text{ch}(r)(M)/r(N) \\ \downarrow f & & \searrow h \\ E(N/r(N)) & & \end{array}$$

Now r is a radical and $N/r(N) \in \mathcal{F}_r$ implies $E(N/r(N)) \in \mathcal{F}_{\text{ch}(r)}$. On the other hand $N/r(N) = h(N/r(N)) \subseteq h(\text{ch}(r)(M)/r(N)) = h(\text{ch}(r)(M/r(N))) \subseteq \text{ch}(r)(E(N/r(N))) = 0$. Thus $N \in \mathcal{F}_r$.

Proposition 1.4.

- (i) Every hereditary preradical is pseudohereditary.
- (ii) Every faithful preradical is pseudohereditary.
- (iii) If r is a cohereditary preradical, then r is pseudohereditary if and only if r is hereditary.
- (iv) If $\text{ch}(r)$ is hereditary, then r is pseudohereditary.
- (v) If R is left hereditary, then r is pseudohereditary implies $\text{ch}(r)$ is so.
- (vi) If $r_i, i \in I$ is a family of preradicals, then $\bigcap_{i \in I} r_i$

is pseudohereditary provided each r_i is so.

(vii) If r is a preradical, then $\bigcap \{s, r \leq s, s \text{ pseudohereditary preradical}\} (\bigcap \{s, r \leq s, s \text{ pseudohereditary radical}\})$ is the least pseudohereditary preradical (pseudohereditary radical) containing r .

(viii) If r is pseudohereditary, then \bar{r} is so.

Proof follows immediately from Definition 1.1 and Proposition 1.2.

The next proposition is a modification of the well known result for hereditary radicals (see Jans [5]).

Proposition 1.5. Let r be a pseudohereditary radical.

Then there is an injective $\text{ch}(r)$ -torsionfree module Q such that $\text{ch}(r) = \text{ch}(p^{\infty}Q)$.

Proof: It is enough to put $Q = \prod_{A \in \mathcal{A}} E(A)$, where \mathcal{A} is a representative set of cyclic r -torsionfree modules. As it is easy to see, Q is an injective $\text{ch}(r)$ -torsionfree module, and therefore $\text{ch}(r) \leq p^{\infty}Q$. On the other hand it suffices to prove $\mathcal{T}_{p^{\infty}Q} \subseteq \mathcal{T}_r$. For, let $T \in \mathcal{T}_{p^{\infty}Q}$, $T \notin \mathcal{T}_r$. Without loss of generality we can assume that $T \in \mathcal{T}_r \cap \mathcal{T}_{p^{\infty}Q}$ (take $T/r(T)$ instead T , if necessary). Therefore T contains a nonzero cyclic submodule C isomorphic to some $A \in \mathcal{A}$. Hence $\text{Hom}_R(C, Q) \neq 0$ and consequently $C \in \mathcal{T}_{p^{\infty}Q}$. On the other side $C \in \mathcal{T}_{p^{\infty}Q}$ since p^{∞} is hereditary, a contradiction.

Corollary 1.6. Let r be a radical. Consider the following conditions:

- (i) r is pseudohereditary,
- (ii) there is an injective module E such that $(0:E) = r(E)$.

Then (i) implies (ii). Moreover, if R is a left hereditary ring then (ii) implies (i).

Proof: (i) implies (ii). By Proposition 1.5.

(ii) implies (i). By Proposition 1.4 (iv), (v).

§ 2. Pseudocohereditary preradicals.

Definition 2.1. A preradical r is said to be pseudocohereditary if for every module M and every epimorphism $M/h(r)(M) \rightarrow A$ $A \in \mathcal{F}_r$.

Proposition 2.2. Let r be a preradical and Q be a faithful injective module. Then the following are equivalent:

- (i) r is pseudocohereditary,
- (ii) $h(r)(M) \subseteq C_r(N:M)$ implies $r(M/N) = (h(r)(M) + N)/N$ for every submodule N of a module M ,
- (iii) If I is an arbitrary index set and $Q^I/r(Q^I) \rightarrow A$ an epimorphism, then $A \in \mathcal{F}_r$.

Proof: (i) implies (ii). Suppose $N \subseteq M$ and $h(r)(M) \subseteq C_r(N:M)$. Consider the natural epimorphism $M/h(r)(M) \rightarrow M/(h(r)(M) + N)$.

According to (i) $(M/N)/((h(r)(M) + N)/N) \cong M/(h(r)(M) + N) \in \mathcal{F}_r$, and hence $r(M/N) \subseteq (h(r)(M) + N)/N$. The converse inclusion is obvious.

(ii) implies (i). If $M \in R\text{-mod}$, $h(r)(M) \subseteq K \subseteq M$ and $M/h(r)(M) \rightarrow M/K$ is a natural epimorphism, then we have $r(M/K) = (h(r)(M) + K)/K = 0$ by (ii).

(i) implies (iii). Obvious.

(iii) implies (i). Let $A, M \in R\text{-mod}$ and $g: M/h(r)(M) \rightarrow A$ be an epimorphism. There is an epimorphism $f: F \rightarrow M$ with F

free. Since Q is faithful $p^{\{Q\}}(F) = 0$, and hence $F \xrightarrow{i} Q^J$ for some index set J . Further, i induces the inclusion $\bar{i}: F/h(r)(F) \rightarrow Q^J/h(r)(Q^J)$. Now consider the push-out diagram

$$\begin{array}{ccc} Q^J/r(Q^J) & \xrightarrow{h} & C \\ \uparrow \bar{i} & & \uparrow j \\ F/h(r)(F) & \xrightarrow{g \circ \bar{f}} & A \end{array}$$

where $\bar{f}: F/h(r)(F) \rightarrow M/h(r)(M)$ is an epimorphism induced by f . As it is easy to see j is a monomorphism and h an epimorphism. According to (iii) $C \in \mathcal{F}_r$, and hence $A \in \mathcal{F}_r$.

Proposition 2.3. Let r be a preradical. Then:

- (i) if r is pseudocohereditary and $0 \rightarrow K \rightarrow A \rightarrow B \rightarrow 0$ is a cover of B , then $B \in \mathcal{T}_r$ implies $A \in \mathcal{T}_{h(r)}$
- (ii) if r is pseudocohereditary and $0 \rightarrow K \hookrightarrow P \rightarrow B \rightarrow 0$ is an arbitrary projective presentation, then $B \in \mathcal{T}_r$ implies $h(r)(P) + K = P$,
- (iii) if R is left perfect, r pseudocohereditary and $C(P) \xrightarrow{\mathcal{G}_P} \mathcal{G}_P \rightarrow P$ a projective cover of P , then $P \in \mathcal{T}_r$ implies $C(P) \in \mathcal{T}_{h(r)}$,
- (iv) if R is left hereditary, r pseudocohereditary and $B \in \mathcal{T}_r$, then there is a projective presentation $0 \rightarrow K \rightarrow P \rightarrow B \rightarrow 0$ with $P \in \mathcal{T}_{h(r)}$,
- (v) if r is an idempotent preradical such that for each $B \in \mathcal{T}_r$ there is a projective presentation $0 \rightarrow K \hookrightarrow P \rightarrow B \rightarrow 0$ with $P = K + h(r)(P)$, then r is pseudocohereditary.

Proof: (i). If $0 \rightarrow K \hookrightarrow A \rightarrow B \rightarrow 0$ is a cover of B and $B \in \mathcal{T}_r$, then $A/K = r(A/K) = (h(r)(A) + K)/K$ implies $A = h(r)(A) + K$, and hence $A \in \mathcal{T}_{h(r)}$, since K is superfluous

in A.

(ii). This can be done in a similar fashion as in (i).

(iii). It follows immediately from (i).

(iv). Let $B \in \mathcal{T}_R$ and $0 \rightarrow K \xrightarrow{f} P \xrightarrow{g} B \rightarrow 0$ be an arbitrary projective presentation. Since r is pseudocohereditary $h(r)(P) + K = P$ due to (ii). Now R is left hereditary and therefore $h(r)(P)$ is projective. Thus $h(r)(P) \in \mathcal{T}_{h(r)}$ and $g(h(r)(P)) = g(P) = B$.

(v). Suppose $N \subseteq M$ and $h(r)(M) \subseteq C_r(N:M)$ and consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \hookrightarrow & P & \xrightarrow{g} & r(M/N) \rightarrow 0 \\ & & & & \searrow f & & \uparrow \sigma \\ & & & & & & C_r(N:M) \end{array}$$

where the row is a projective presentation of $r(M/N)$ such that $K + h(r)(P) = P$ and σ is a natural epimorphism. Now $r(M/N) = g(h(r)(P)) = \sigma(f(h(r)(P))) \subseteq \sigma(h(r)(M)) = (h(r)(M) + N)/N$ and consequently $r(M/N) = (h(r)(M) + N)/N$.

Proposition 2.4.

- (i) Every cohereditary preradical is pseudocohereditary.
- (ii) Every cofaithful preradical is pseudocohereditary.
- (iii) If r is a hereditary preradical, then r is pseudocohereditary if and only if r is cohereditary.
- (iv) If $h(r)$ is cohereditary, then r is pseudocohereditary.
- (v) If R is left hereditary, and r a pseudocohereditary preradical, then $h(r)$ is cohereditary.
- (vi) If $r_i, i \in I$ is a family of preradicals, then $\sum_{i \in I} r_i$ is pseudocohereditary provided each r_i is so.

(vii) If r is a preradical, then $\sum \{s, s \leq r, s\text{-pseudocohereditary preradical}\}$ ($\sum \{s, s \leq r, s\text{-pseudocohereditary idempotent preradical}\}$) is the largest pseudocohereditary (pseudocohereditary idempotent) preradical contained in r .

(viii) If r is pseudocohereditary, then \tilde{r} is so.

Proof follows immediately from Definition 2.1 and Proposition 2.2.

Proposition 2.5. Let R be either left hereditary or left perfect and r be a pseudocohereditary idempotent preradical. Then $h(r) = h(p_{\{P\}})$ for some $h(r)$ -torsion projective module P .

Proof: Let \mathcal{Q} be a representative set of cocyclic r -torsion modules and P be the direct sum of projective $h(r)$ -torsion presentations of modules from \mathcal{Q} (the existence of P follows from Proposition 2.3(iii), (iv)). As it is easy to see P is a projective $h(r)$ -torsion module, and therefore $p_{\{P\}} \leq h(r)$. On the other hand it suffices to show that $\mathcal{F}_{p_{\{P\}}} \subseteq \mathcal{F}_r$. For, let $F \in \mathcal{F}_{p_{\{P\}}}$ and $F \notin \mathcal{F}_r$. Without loss of generality we can assume that $F \in \mathcal{F}_r \cap \mathcal{F}_{p_{\{P\}}}$ (take $r(F)$ instead F , if necessary). If C is a nonzero cocyclic factormodule of F , then $C \cong A$ for some $A \in \mathcal{Q}$. Hence $\text{Hom}_R(P, C) \neq 0$ and $C \notin \mathcal{F}_{p_{\{P\}}}$. On the other hand $C \in \mathcal{F}_{p_{\{P\}}}$ since $p_{\{P\}}$ is cohereditary, a contradiction.

Corollary 2.6. Let r be an idempotent preradical for R -mod, where R is a left hereditary ring. Then the following are equivalent:

- (i) r is pseudocohereditary,

(ii) there is a projective module P such that $r(M) = p_{\{P\}}(M)$ for every injective module M .

Proof: (i) implies (ii). By Proposition 2.5.

(ii) implies (i). By Proposition 2.4(iv),(v).

R e f e r e n c e s

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(Oblatum 26.2.1979)

