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BEHAVIOUR OF MACHINES IN CATEGORIES

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Abstract: Functorial machines in the category Set of sets are introduced such that they include Arbib Manes machines in Set and Eilenberg's X-machines. Their behaviour is introduced as the smallest solution of a suitable equation and the coincidence of the usual notion of the behaviour is proved.

Key words: Category, functor, relation, machine, automaton, functorial algebra, behaviour.

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In [E], S. Eilenberg introduces a notion of X-machines and the relation computed by it. He unifies the description of the action of two ways automata, push-down automata, Turing machines and, as he says, "the list of examples could be continued indefinitely ([E, p. 288]). In [AM], M.A. Arbib and E.G. Manes define functorial machines in a category to unify the description of sequential automata, tree automata and others. In the present paper, we define functorial machines and their behaviour and show that this makes it possible to describe the above X-machines of [E] and Arbib Manes functorial machines and their action in a unified way. The smallest-solution-technique is used here in a general functorial form. To keep the formal apparatus simple, we deal with the category Set of all sets only. Some generalizations are sketched at the end of the paper.

I. <u>Machines and their behaviour</u>

1. Denote by Set the category of all sets and all their mappings and by Rel the category of all sets and all their (binary) relations, no matter whether a binary relation $r: A \twoheadrightarrow B$ is supposed to be a mapping of A into the set of all subsets of B or to be an ordered triple (A,C,B), where $C \subset A \times B$ or to be the ordered pair (π_A, π_B), where $\pi_A : C \longrightarrow A$, $\pi_B : C \longrightarrow B$ are the projections; any of the three forms of the description will be used. Moreover, if $\alpha : X \longrightarrow A$, $\beta : X \longrightarrow B$ are mappings, we denote by $[\alpha, \beta]$ the relation $(A, \{(\alpha(x), \beta(x)) \mid x \in X\}, B)$. (Let us indicate by $A \longrightarrow B$ a mapping and by $A \twoheadrightarrow B$ a relation; $\cdot$ denotes the composition of mappings and $\circ$ the composition of relations.)

2. If $r_i : A \twoheadrightarrow B$ are relations, $r_i = (A, C_i, B)$, we define, as usual,

$r_1 \leq r_2$ iff $C_1 \subset C_2$,

$r_1 + r_2 = (A, C_1 \cup C_2, B)$ (more generally, $\sum_i r_i = (A, \bigcup_i C_i, B)$,

$r_i^{-1} = (B, C_i^{-1}, A)$.

3. Let $F: \mathrm{Set} \longrightarrow \mathrm{Set}$ be a functor. A relational F-algebra is any pair (Q, δ), where Q is a set and $\delta : FQ \twoheadrightarrow Q$ is a relation. If δ is a mapping then (Q, δ) is called only F-algebra. A homomorphism $h: (Q, \delta) \longrightarrow (Q', \delta')$ of F-algebras is every mapping $h: Q \longrightarrow Q'$ such that $\delta \cdot h = F(h) \cdot \delta'$. <u>A free</u>

F-algebra over a set I consists of an F-algebra $(I^{\#}, \varphi)$ and a mapping $\eta : I \longrightarrow I^{\#}$ with the following universal property: for every F-algebra (Q, δ) and every mapping $i : I \longrightarrow Q$ there exists a unique homomorphism $i^{\#} : (I^{\#}, \varphi) \longrightarrow (Q, \delta)$ such that $\eta \cdot i^{\#} = i$. The mapping $i^{\#}$ is called a free extension of i (with respect to δ) [AM].

A functor F:Set $\longrightarrow$ Set for which a free F-algebra exists over any set I is called a varietor. All varietors in Set were characterized in [KK].

4. Let F:Set $\longrightarrow$ Set be a functor. We extend it to a mapping $\bar{F}$:Rel $\longrightarrow$ Rel by the rule

$$F[\alpha, \beta] = [F(\alpha), F(\beta)].$$

If $[\alpha_1, \beta_1] = [\alpha_2, \beta_2]$, then $[F(\alpha_1), F(\beta_1)] = [F(\alpha_2), F(\beta_2)]$ For, put $\{(\alpha_1(x), \beta_1(x)) \mid x \in X_1\} = C = \{(\alpha_2(x), \beta_2(x)) \mid x \in X_2\}$ and denote by $\pi_A : C \longrightarrow A$, $\pi_B : C \longrightarrow B$ the projections. Then $\rho_i \cdot \pi_A = \alpha_i$, $\rho_i \cdot \pi_B = \beta_i$ for a surjective mapping $\rho_i : X_i \longrightarrow C$, $i = 1,2$. Since ρ_1, ρ_2 are retractions, $F(\rho_1)$ and $F(\rho_2)$ are also surjective. Hence $[F(\alpha_1), F(\beta_1)] =$ $= [F(\rho_1) \cdot F(\pi_A), F(\rho_1) \cdot F(\pi_B)] = [F(\pi_A), F(\pi_B)] =$ $= [F(\rho_2) \cdot F(\pi_A), F(\rho_2) \cdot F(\pi_B)] = [F(\alpha_2), F(\beta_2)]$. The mapping F:Rel $\longrightarrow$ Rel has the following properties:

1) $\bar{F}(r_1 \circ r_2) \leq \bar{F}(r_1) \circ \bar{F}(r_2)$;

2) if $r_1 \leq r_2$, then $\bar{F}(r_1) \leq \bar{F}(r_2)$;

3) $\bar{F}(r^{-1}) = (\bar{F}(r))^{-1}$.

In [T$_1$], all the functors $\bar{F}$:Set $\longrightarrow$ Set, for which the extension $\bar{F}$:Rel $\longrightarrow$ Rel satisfies the stronger condition

1´) $$\bar{F}(r_1 \circ r_2) = \bar{F}(r_1) \circ F(r_2)$$

(i.e. F is an endofunctor of Rel) are characterized. Since we need this in II., we recall the characterization. We say that $F:\mathrm{Set} \longrightarrow \mathrm{Set}$ covers pullbacks if, for every pullbacks

$$\begin{tikzcd}[column sep=small] & \bullet \arrow[dr, "\alpha_1"] & \\ \bullet \arrow[ur, "\bar{\alpha}_2"] \arrow[dr, "\bar{\alpha}_1"'] & & \circ \\ & \bullet \arrow[ur, "\alpha_2"'] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}[column sep=small] & \bullet \arrow[dr, "F\alpha_1"] & \\ \bullet \arrow[ur, "\tilde{\alpha}_2"] \arrow[dr, "\tilde{\alpha}_1"'] & & \bullet \\ & \bullet \arrow[ur, "F\alpha_2"'] & \end{tikzcd}$$

the unique mapping φ which fulfils $\varphi \cdot \tilde{\alpha}_i = F(\bar{\alpha}_i)$, i = = 1,2, is surjective.

<u>Proposition</u> $[T_1]$: $\bar{F}:\mathrm{Rel} \longrightarrow \mathrm{Rel}$ is an endofunctor iff F covers pullbacks.

5. Let $F:\mathrm{Set} \longrightarrow \mathrm{Set}$ be a functor. Let us denote by the same letter $F:\mathrm{Rel} \longrightarrow \mathrm{Rel}$ its extension as in 4.

<u>An</u> <u>F-machine</u> $\mathbb{M}$ <u>in</u> Set consists of the following data. Two-relational F-algebras, say

(J, ψ) ... called <u>the type algebra of</u> $\mathbb{M}$ and

(Q, δ) ... called <u>the state algebra of</u> $\mathbb{M}$ and three relations situated as follows.

$\alpha : A \twoheadrightarrow J$ called <u>the input code of</u> $\mathbb{M}$,

$\iota : J \twoheadrightarrow Q$ called <u>the initiation of</u> $\mathbb{M}$,

$y : Q \twoheadrightarrow Y$ called <u>the output of</u> $\mathbb{M}$.

The situation is visualized on the picture below.

$$\begin{tikzcd} & FJ \arrow[d, "\psi"] & & FQ \arrow[d, "\delta"'] & \\ A \arrow[r, "\alpha"', two heads] & J \arrow[rr, "\iota"', two heads] & & Q \arrow[r, "y"', two heads] & Y \end{tikzcd}$$

We write $\mathbb{M} = \{\alpha, (J, \psi), \iota, (Q, \delta), y\}$.

6. <u>The run</u> $\iota^* : J \twoheadrightarrow Q$ of a machine $\mathbb{M} = \{\alpha, (J, \psi), \iota, (Q, \sigma), y\}$ is defined as the smallest solution of the equation

$$x = \iota + \psi^{-1} \circ F(x) \circ \sigma .$$

<u>The behaviour of</u> $\mathbb{M}$ is defined by

$$\text{beh } \mathbb{M} = \alpha \circ \iota^* \circ y .$$

7. <u>The run construction</u>. Let $\mathbb{M} = \{\alpha, (J, \psi), \iota, (Q, \sigma), y\}$ be an F-machine. We define by induction over all ordinals

$$r_0 = \iota ,$$

$$r_{\alpha+1} = \iota + \psi^{-1} \circ F(r_\alpha) \circ \sigma ,$$

$$r = \sum_{\beta < \alpha} r_\beta \quad \text{for } \alpha \text{ limit ordinal.}$$

We say that the run construction stops (after γ steps) if $r_\gamma = r_{\gamma+1}$. Then $r_{\gamma'} = r_\gamma$ for all $\gamma' \geq \gamma$.

<u>Lemma</u>. If $\alpha \leq \alpha'$, then $r_\alpha \leq r_{\alpha'}$.

<u>Proof</u> by induction.

<u>Corollary</u>. The run construction always stops, at most after card $(J \times Q)$ steps, no matter what the functor F is.

<u>Proposition</u>. If $r_\gamma = r_{\gamma+1}$, then $r_\gamma = \iota^*$ is the run of $\mathbb{M}$.

<u>Proof</u>. If $r_\gamma = r_{\gamma+1}$, then r_γ is a solution of the equation $x = \iota + \psi^{-1} \circ F(x) \circ \sigma$, evidently. Let $\sigma : J \twoheadrightarrow Q$ be a relation such that $\sigma = \iota + \psi^{-1} \circ F(\sigma) \circ \sigma$. Then $r_\alpha \leq \sigma$ for all ordinals α (the straightforward proof by induction is omitted) hence $\iota^* \leq \sigma$. Thus, ι^* is the smallest solution of the equation.

8. Let $\mathbb{M} = \{\alpha, (J, \psi), \iota, (Q, \sigma), y\}$ be a machine.

__A reversed machine__ $\mathbb{M}^{-1}$ is defined to be $[\![y^{-1}, (Q,\sigma), \iota^{-1}, (J,\psi), \alpha^{-1}]\!]$.

__Observation__: run $\mathbb{M}^{-1} = (\text{run } \mathbb{M})^{-1}$,

beh $\mathbb{M}^{-1} = (\text{beh } \mathbb{M})^{-1}$.

9. A machine $\mathbb{M} = [\![\alpha, (J,\psi), \iota, (Q,\sigma), y]\!]$ is called __standard__ if $\psi : FJ \longrightarrow J$ is a mapping.

__Proposition__. Let $\mathbb{M} = [\![\alpha, (J,\psi), \iota, (Q,\sigma), y]\!]$ be a standard machine. Then its run ι^* is the smallest relation $J \longrightarrow\!\!\!\!\rightarrow Q$ such that

$$\psi \circ \iota^* \geq F(\iota^*) \circ \sigma ,$$
$$\iota^* \geq \iota .$$

__Proof__. First, let us notice that if $\psi : FJ \longrightarrow J$ is a mapping, then $\psi \circ \psi^{-1} \geq 1_{FJ}$, $\psi^{-1} \circ \psi \leq 1_J$.

a) The run ι^* is the smallest solution of the equation $x = \iota + \psi^{-1} \circ F(x) \circ \sigma$. Hence $\iota^* \geq \iota$ and $\psi \circ \iota^* = \psi \circ (\iota + \psi^{-1} \circ F(\iota^*) \circ \sigma) = \psi \circ \iota + \psi \circ \psi^{-1} \circ F(\iota^*) \circ \sigma \geq F(\iota^*) \circ \sigma$.

b) Let ϱ be a relation $J \longrightarrow\!\!\!\!\rightarrow Q$ such that $\psi \circ \varrho \geq F(\varrho) \circ \sigma$ and $\varrho \geq \iota$. We show $r_\alpha \leq \varrho$ for all ordinals α, by induction. Clearly $\iota = r_0 \leq \varrho$. If $r_\alpha \leq \varrho$, then $r_{\alpha+1} = \iota + \psi^{-1} \circ F(r_\alpha) \circ \sigma \leq \iota + \psi^{-1} \circ F(\varrho) \circ \sigma \leq \iota + \psi^{-1} \circ \psi \circ \varrho \leq \iota + \varrho \leq \varrho$. If $r_\beta \leq \varrho$ for all $\beta < \alpha$, then $\sum_{\beta < \alpha} r_\beta \leq \varrho$. We conclude that $\iota^* \leq \varrho$.

__Remark__. In $[T_1], [T_2]$ the run of a machine is defined as the smallest relation which fulfils the above inequalities. As it is proved, this coincides with our definition of run for standard machines, but not in general.

10. Let $F : \text{Set} \longrightarrow \text{Set}$ be a varietor (see 3.). We say that

an F-machine $\mathbb{M} = [\![\alpha, (J, \psi), \iota, (Q, \delta), y]\!]$ is <u>a free machine</u> if its input code α is the identity 1_J, its type algebra (J, ψ) is a free F-algebra over a set I and its initiation ι factors through $[\eta, 1_I]$ where $\eta : I \to I^{\#}$ is the universal mapping of the free F-algebra $(I^{\#}, \varphi) = (J, \psi)$ (see 3.). Free machines coincide with relational automata, investigated in $[T_1]$. We say that $\mathbb{M}$ is <u>a free deterministic machine</u> if it is a free machine such that $\delta : FQ \to Q$ and $y: Q \to Y$ are mappings and $\iota = [\eta, i]$, where $i : I \to Q$ is a mapping. Free deterministic machines coincide with the Arbib-Manes machines in the category Set, see [AM]. The definition of behaviour also coincides (in [AM], the behaviour is defined to be $i^{\#} \cdot y : I^{\#} \to Y$, where $i^{\#}$ is the free extension of $i : I \to Q$). This follows from the proposition below.

<u>Proposition</u>. Let $\mathbb{M} = [\![1_{I^{\#}}, (I^{\#}, \varphi), [\eta, i], (Q, \delta), y]\!]$ be a free deterministic machine. Then its run ι^{*} is the free extension $i^{\#}$ of i.

<u>Proof</u>. Since every free machine is a standard one, it is sufficient to prove that the free extension $i^{\#}$ is the smallest relation $I^{\#} \twoheadrightarrow Q$ which fulfils $\varphi \circ i^{\#} \geq F(i^{\#}) \circ \delta$ and $i^{\#} \geq [\eta, i]$. Clearly, $i^{\#}$ really fulfils the inequalities. Now, let $r : I^{\#} \twoheadrightarrow Q$ be a relation such that $\varphi \circ r \geq F(r) \circ \delta$ and $r \geq [\eta, i]$. Let $r = (I^{\#}, C, Q)$, let $\alpha : C \to I^{\#}$, $\beta : C \to Q$ be projections. Let $\varphi, \alpha, \tilde{\varphi}, \tilde{\alpha}$ form a pullback ($\tilde{\varphi}$ opposite to φ, $\tilde{\alpha}$ opposite to α). Denote by X the common domain of $\tilde{\alpha}$ and $\tilde{\varphi} \cdot \beta$. Then $\varphi \circ r = [\tilde{\alpha}, \tilde{\varphi} \cdot \beta]$ and, since X is the preimage of C in the mapping $\varphi \times 1_Q$, $\tilde{\alpha} : X \to FJ$, $\tilde{\varphi} \cdot \beta : X \to Q$ are projections again. Since $\varphi \circ r \geq$

$\geq F(r) \circ \sigma$, there exists a mapping $\rho : F(C) \to X$ such that $\rho \cdot \tilde{\alpha} = F(\alpha)$, $\rho \cdot \tilde{\varphi} \cdot \beta = F(\beta) \cdot \sigma$. Since $r \geq [\eta, i]$, there exists a mapping $\gamma : I \to C$ such that $\gamma \cdot \alpha = \eta$, $\gamma \cdot \beta = i$. Consider the F-algebra $(C, \rho \cdot \tilde{\varphi})$. Denote by $\gamma^{*} : (I^{\#}, \varphi) \to (C, \rho \cdot \tilde{\varphi})$ the free extension of γ . Since $\rho \cdot \tilde{\varphi} \cdot \alpha = \rho \cdot \tilde{\alpha} \cdot \varphi = F(\alpha) \cdot \varphi$, we conclude that $\alpha : (C, \rho \cdot \tilde{\varphi}) \to (I^{\#}, \varphi)$ is a homomorphism. Since $\gamma^{\#} \cdot \alpha$ is a homomorphism of $(I^{\#}, \varphi)$ into itself and $\eta \cdot (\gamma^{\#} \cdot \alpha) = \gamma \cdot \alpha = \eta$, $\gamma^{\#} \cdot \alpha$ must be $1_{I^{\#}}$. Since $\beta : (C, \rho \cdot \tilde{\varphi}) \to (Q, \sigma)$ is a homomorphism and $\eta \cdot \gamma^{\#} \cdot \beta = i$, the mapping $\gamma^{\#} \cdot \beta$ is equal to $i^{\#}$. We conclude that $i^{\#} = [1_{I^{\#}}, i^{\#}] = [\gamma^{\#} \cdot \alpha, \gamma^{\#} \cdot \beta] \leq [\alpha, \beta]$.

<u>Note</u>. The above proof could be simplified for Set, but we preferred the form which works for general categories without any modification.

II. <u>Free components of machines</u>

1. Let $F : \mathrm{Set} \to \mathrm{Set}$ be a varietor. Let

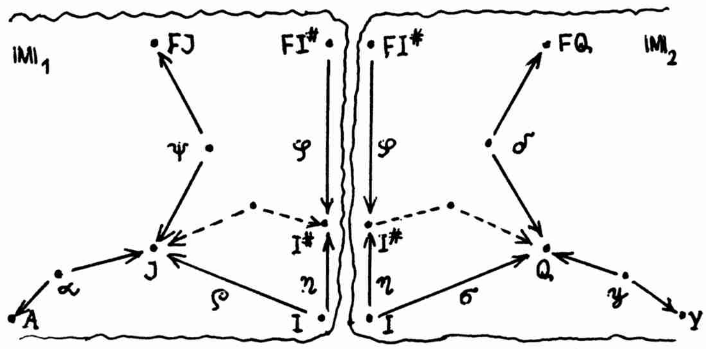
$$\mathbb{M} = [\![\alpha, (J, \psi), \iota, (Q, \sigma), y]\!]$$

be an F-machine. Let its initiation be expressed as $\iota = (J, I, Q)$, $I \subset J \times Q$, let $\rho : I \to J$, $\sigma : I \to Q$ be the projections. Let $(I^{\#}, \varphi)$ and $\eta : I \to I^{\#}$ form the free F-algebra over the set I. We define <u>free components of</u> $\mathbb{M}$ (the first $\mathbb{M}_1$ and <u>the second</u> $\mathbb{M}_2$) as

$$\mathbb{M}_1 = [\![1_{I^{\#}}, (I^{\#}, \varphi), [\eta, \rho], (J, \psi), \alpha^{-1}]\!],$$

$$\mathbb{M}_2 = [\![1_{I^{\#}}, (I^{\#}, \varphi), [\eta, \sigma], (Q, \sigma), y]\!].$$

Clearly, $\mathbb{M}_1$ and $\mathbb{M}_2$ are free machines. $\mathbb{M}_1$ is deterministic iff $\mathbb{M}$ is standard. $\mathbb{M}_2$ is deterministic iff $\mathbb{M}^{-1}$ is standard. The situation is visualized on the following picture.



2. Let $F:\mathrm{Set} \longrightarrow \mathrm{Set}$ be a varietor, let $\mathbb{M}$ be an F-machine. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be its first and the second free components.

<u>Proposition</u>. $\mathrm{run}\ \mathbb{M} \leq (\mathrm{run}\ \mathbb{M}_1)^{-1} \circ \mathrm{run}\ \mathbb{M}_2$. If either $\mathbb{M}$ or $\mathbb{M}^{-1}$ is standard or if F covers pullbacks, then

$\mathrm{run}\ \mathbb{M} = (\mathrm{run}\ \mathbb{M}_1)^{-1} \circ \mathrm{run}\ \mathbb{M}_2$ and

$\mathrm{beh}\ \mathbb{M} = (\mathrm{beh}\ \mathbb{M}_1)^{-1} \circ \mathrm{beh}\ \mathbb{M}_2$.

<u>Proof</u>. Let us apply the run construction on $\mathbb{M}_1$, $\mathbb{M}_2$ and $\mathbb{M}_3 = \mathbb{M}$. Denote the corresponding r_α's by $r_{i,\alpha}$, i = = 1,2,3. Clearly, $r_{3,0} = r_{1,0}^{-1} \circ r_{2,0}$. If $r_{3,\alpha} \leq r_{1,\alpha}^{-1} \circ r_{2,\alpha}$, then $r_{3,\alpha+1} = r_{3,0} + \psi^{-1} \circ F(r_{3,\alpha}) \circ \delta \leq r_{1,0}^{-1} \circ r_{2,0} + {}$ $+ \psi^{-1} \circ F(r_{1,\alpha}^{-1}) \circ \varphi \circ \varphi^{-1} \circ F(r_{2,\alpha}) \circ \delta = r_{1,\alpha+1}^{-1} \circ r_{2,\alpha+1}$ (the last equality is based on the fact that $I^{\#}$ is a coproduct of I and $FI^{\#}$ with the coproduct-injections $\eta : I \longrightarrow I^{\#}$, $\varphi : FI^{\#} \longrightarrow I^{\#}$, hence the relations $\eta \circ \varphi^{-1}$ and $\varphi \circ \eta^{-1}$ are empty). The limit step is evident. We conclude that $\mathrm{run}\ \mathbb{M} \leq$ $\leq (\mathrm{run}\ \mathbb{M}_1)^{-1} \circ \mathrm{run}\ \mathbb{M}_2$. If either $\mathbb{M}$ or $\mathbb{M}^{-1}$ is standard or if F covers pullbacks (see I.4.), then always $F(r_{1,\alpha}^{-1}) \circ$ $\circ F(r_{2,\alpha}) = F(r_{1,\alpha}^{-1} \circ r_{2,\alpha})$. This makes it possible to show that $r_{3,\alpha} = r_{1,\alpha}^{-1} \circ r_{2,\alpha}$ for all α , so $\mathrm{run}\ \mathbb{M} = (\mathrm{run}\ \mathbb{M}_1)^{-1} \circ$

$\circ$ run $\mathbb{M}_2$. The second equation concerning beh $\mathbb{M}$ is an immediate consequence of the first one.

3. Let us say that a pullback

$$\begin{tikzcd} & \bullet \arrow[dr, "f"] & \\ \bullet \arrow[ur] \arrow[dr] & & \bullet \\ & \bullet \arrow[ur, "g"'] & \end{tikzcd}$$

is the pullback formed by f and g. We say that $F:\mathrm{Set}\longrightarrow\mathrm{Set}$ preserves preimages if the F-image of every pullback formed by a pair of mappings f, g with f one-to-one, is a pullback again. By [T_1] if F covers pullbacks, then it preserves preimages.

<u>Proposition</u>. Let $F:\mathrm{Set}\longrightarrow\mathrm{Set}$ be a preimage preserving varietor. Then the equation

$$\mathrm{beh}\ \mathbb{M} = (\mathrm{beh}\ \mathbb{M}_1)^{-1}\circ \mathrm{beh}\ \mathbb{M}_2$$

holds for every F-machine $\mathbb{M}$ (with $\mathbb{M}_1$ and $\mathbb{M}_2$ being the free components of $\mathbb{M}$) if and only if F covers pullbacks.

<u>Proof</u>. By 2., we have only to show that if F does not cover pullbacks, then there exists an F-machine $\mathbb{M}$ with beh $\mathbb{M} \neq (\mathrm{beh}\ \mathbb{M}_1)^{-1}\circ \mathrm{beh}\ \mathbb{M}_2$. It will be shown in several steps.

a) Since F does not cover pullbacks, it is not a constant functor. Denote by $F\emptyset = D$. Then we may suppose (up to natural equivalence) that $D\subset FX$ for every set X and $(Ff)(d) = d$ for every mapping f and every $d\in D$. Since F is supposed to preserve preimages, we have

$$(Ff)(FX)\cap(Fg)(FY) = D$$

for every pair of mappings $f:X\longrightarrow A$, $g:Y\longrightarrow A$ with $f(X)\cap g(X)= \emptyset$ and f being one-to-one.

b) Lemma. Let there exist a cardinal m such that card $(FX \setminus D) \leq m$ for all sets X. Then F is a constant functor.

Proof. By [K], if card $FX <$ card X for some set X, then F is constant up to X.

c) Lemma. Let F do not cover pullbacks. Then there exists a non-empty set L and mappings $\mu_i : FL \longrightarrow FL$, $i = 1,2$, such that $\mu_i(d) = d$ for all $d \in D$ and F does not cover the pullback formed by μ_1 and μ_2.

Proof. Since F does not cover pullbacks, there exist mappings $f_1 : A_1 \longrightarrow A_3$, $f_2 : A_2 \longrightarrow A_3$ such that F does not cover the pullback formed by f_1 and f_2. Put $m = \aleph_0 \cdot \max_{j=1,2,3} \text{card } A_j$. Then F does not cover the pullback formed by $1_m \amalg f_1$ and $1_m \amalg f_2$ (where $\amalg$ denotes a coproduct in Set). Denote $f_i' = 1_m \amalg f_i$, $i = 1,2$, $A_j' = m \amalg A_j$, $j = 1,2,3$. By the choice of m we obtain card $A_j' = m$ for $j = 1,2,3$. Find a non-empty set L such that card $(FL \setminus D) \geq m$ (this is possible, by b)) and choose one-to-one mappings $\gamma_j : A_j' \longrightarrow FL \setminus D$ such that $FL \setminus (D \cup \gamma_j(A_j'))$ have the same cardinality for $j=1,2,3$. Choose a bijection σ_i of $FL \setminus \gamma_i(A_i')$ onto $FL \setminus \gamma_3(A_3')$, identical on D, $i=1,2$, and define $\mu_i : FL \longrightarrow FL$ as $\gamma_i^{-1} \circ f_i' \circ \gamma_3$ on $\gamma_i(A_i')$ and σ_i on $FL \setminus \gamma_i(A_i')$. Then F does not cover the pullback formed by μ_1 and μ_2.

d) Now, we finish the proof of the proposition. Let L and $\mu_i : FL \longrightarrow FL$ be as in c). Denote by $\varepsilon_1 : L \longrightarrow L \amalg FL$ and $\varepsilon_2 : FL \longrightarrow L \amalg FL$ the coproduct injections. Put

$$M = L \amalg F(L \amalg FL)$$

and denote by $e_1 : L \longrightarrow M$ the first coproduct injection $v : F(L \amalg FL) \longrightarrow M$ the second coproduct injection and put

$$(F\varepsilon_1) \cdot v = e_2 : FL \longrightarrow M, \quad (F\varepsilon_2) \cdot v = e_3 : FFL \longrightarrow M.$$

We have $(F\varepsilon_1)(FL) \cap (F\varepsilon_2)(FFL) = D$. Define $q_i : FM \twoheadrightarrow M$ by

$q_i = [\mu_i \cdot Fe_1, e_2] + [Fe_2, e_3]$. We define a machine $\mathbb{M}$ as follows:

$\mathbb{M} = \langle 1, (M, q_1), [e_1, e_1], (M, q_2), 1 \rangle$.

We show that run $\mathbb{M} \neq (\text{run } \mathbb{M}_1)^{-1} \circ \text{run } \mathbb{M}_2$. Denote by ι^*_i the run of $\mathbb{M}_i$, $i = 1,2,3$ ($\mathbb{M}_3 = \mathbb{M}$). Then $e_1 \circ \iota^*_3 \circ e_1^{-1} = 1_\iota$ and $e_2 \circ \iota^*_3 \circ e_2^{-1} = e_2 \circ [e_1, e_1] \circ e_2^{-1} + e_2 \circ e_2^{-1} \circ \mu_1 \circ Fe_1 \circ F\iota^*_3 \circ Fe_1^{-1} \circ \mu_2^{-1} \circ e_2 \circ e_2^{-1}$.

Since the first summand is $\emptyset$ and since $Fe_1 \circ F\iota^*_3 \circ Fe_1^{-1} = F(e_1 \circ \iota^*_3 \circ e_1^{-1})$ (because F preserves preimages), we obtain

$$e_2 \circ \iota^*_3 \circ e_2^{-1} = \mu_1 \circ F(e_1 \circ \iota^*_3 \circ e_1^{-1}) \circ \mu_2^{-1} = \mu_1 \circ \mu_2^{-1}$$

$$e_3 \circ \iota^*_3 \circ e_3^{-1} = e_3 \circ e_3^{-1} \circ Fe_2 \circ F\iota^*_3 \circ Fe_2^{-1} \circ e_3 \circ e_3^{-1} =$$
$$= F(e_2 \circ \iota^*_3 \circ e_2^{-1}) = F(\mu_1 \circ \mu_2^{-1}).$$

One can prove analogously that $e_3 \circ (\iota^*_1)^{-1} \circ \iota^*_2 \circ e_3^{-1} = F\mu_1 \circ F\mu_2^{-1}$. Since F does not cover the pullback formed by μ_1 and μ_2, we conclude that $\iota^*_3 \neq (\iota^*_1)^{-1} \circ \iota^*_2$.

<u>Problem</u>. Does the above proposition hold without the assumption that F preserves preimages?

4. <u>Examples</u>. Let Ω be a type, i.e. a set endowed with an arity function $ar: \Omega \to \{\text{cardinals}\}$. The functor F_Ω : :Set $\to$ Set is defined by

$$F_\Omega X = \coprod_{\omega \in \Omega} X^{ar(\omega)}, \quad F_\Omega f = \coprod_{\omega \in \Omega} f^{ar(\omega)}.$$

As it is well-known, F_Ω preserves pullbacks for every Ω and every arity function, so it covers pullbacks. Denote by P: :Set $\to$ Set the covariant power-set functor, i.e.

$$PX = \{Z \subset X\}, \text{ Pf sends Z to } f(Z).$$

For any cardinal m, denote by P_m:Set $\to$ Set its subfunctor defined by

$$P_m X = \{ Z \subset X \mid \text{card } Z \le m \}.$$

All the functors P, P_m, $m \in \{\text{cardinals}\}$, preserve preimages. P covers pullbacks (but it does not preserve them), but

P_m covers pullbacks iff either $m < 3$ or $m \ge \aleph_0$.

(For example, P_3 does not cover the pullback formed by $f: \{0,1,2\} \to \{0,1\}$ and $g: \{0,1,2\} \to \{0,1\}$, where $f(0) = f(1) = 0$, $f(2) = 1$, $g(0) = 0$, $g(1) = g(2) = 1$.)

Hence, by 3., there exists a P_3-machine $|M|$ with run $|M| < (\text{run } |M|_1)^{-1} \circ \text{run } |M|_2$. On the other hand, there exists no such F-machine with either $F = F_\Omega$ or $F = P$ or $F = P_m$ with $m < 3$ or $m \ge \aleph_0$.

III. Relations computed by X-machines

1. Let us recall (with formal modifications) the notion of an X-machine in the sense of Eilenberg [E, p. 267]. An X-machine $\mathcal{M}$ over an alphabet Σ consists of the following data.

a) A finite Σ-automaton $\mathcal{A} = (Q, I, T)$ (i.e. a finite set Q of states, $I \subset Q$ initial states, $T \subset Q$ terminal states) with a next state relation $\delta: Q \times \Sigma \twoheadrightarrow Q$;

b) a relation $\varphi: X \times \Sigma \twoheadrightarrow X$;

c) an input code $\alpha: A \twoheadrightarrow X$ and an output code $\omega: X \twoheadrightarrow Y$.

For every $\sigma \in \Sigma$, let us denote $\varphi(-, \sigma): X \twoheadrightarrow X$ by R_σ and $\delta(-, \sigma): Q \twoheadrightarrow Q$ by D_σ. The relation $|\mathcal{M}|: X \twoheadrightarrow X$ is defined in [E] as $\bigcup R_{\sigma_1} \circ \dots \circ R_{\sigma_n}$, where the union is taken over all strings $\sigma_1 \dots \sigma_n$ accepted by the automaton $\mathcal{A}$. The relation computed by $\mathcal{M}$ is defined as $\alpha \circ |\mathcal{M}| \circ \omega$.

Define $F_\Sigma: \text{Set} \to \text{Set}$ by $F_\Sigma A = A \times \Sigma$, $F_\Sigma f = f \times 1_\Sigma$.

For every X-machine $\mathcal{M}$ define an F_Σ -machine $\mathbb{M}(\mathcal{M})$ as follows.

$\mathbb{M}(\mathcal{M}) = [\![\alpha, (X,\pi), [p, 1_X\times i], (X\times Q, \lambda), [1_X\times t, \bar{p}]\circ\omega]\!]$, where $i: I \longrightarrow Q$, $t: T \longrightarrow Q$ are inclusions; $\pi: X\times\Sigma \longrightarrow X$, $p: X\times I \longrightarrow X$, $\bar{p}: X\times T \longrightarrow X$ are the first projections and $\lambda(-,-,\sigma) = R_\sigma \times D_\sigma : X\times Q \longrightarrow\!\!\!\!\rightarrow X\times Q$. The situation is visualized on the picture below.

$$\begin{tikzcd}
& X\times\Sigma \arrow[d, "\pi"] & & X\times Q\times\Sigma \arrow[d, "\lambda"] & & & \\
& X & & X\times Q & & X \arrow[r, two heads, "\omega"] & Y \\
A \arrow[ur, "\alpha"'] & & X\times I \arrow[ul, "p"'] \arrow[ur, "1_X\times i"] & & X\times T \arrow[ul, "1_X\times t"'] \arrow[ur, "\bar{p}"] & &
\end{tikzcd}$$

2. <u>Proposition</u>. The relation computed by $\mathcal{M}$ is equal to beh $\mathbb{M}(\mathcal{M})$.

<u>Proof</u>. We consider the free components of $\mathbb{M}(\mathcal{M})$ (see II.1). Denote by Σ^* the free monoid over Σ and by Λ the empty string. The free F_Σ -algebra over $X\times I$ is formed by $(X\times I\times\Sigma^*, \varphi)$ and $\eta: X\times I \longrightarrow X\times I\times\Sigma^*$, where $\varphi: X\times I\times\Sigma^*\times\Sigma \longrightarrow X\times I\times\Sigma^*$ sends every (x,q,s,σ) to $(x,q,s\sigma)$ and η sends (x,q) to (x,q,Λ). The free extension $p^\#: (X\times I\times\Sigma^*, \varphi) \longrightarrow (X,\pi)$ sends every (x,q,s) to x while the free extension $(1_X\times i)^\#: (X\times I\times\Sigma^*, \varphi) \longrightarrow (X\times Q, \lambda)$ sends every (x,q,s) with $s = \sigma_1\dots\sigma_n$ to $(R_{\sigma_1}\circ\dots\circ R_{\sigma_n}(x))\times(D_{\sigma_1}\circ\dots\circ D_{\sigma_n}(x))$. Hence

$$X\times Q\times\Sigma^* \xrightarrow{(1_X\times i)^\#} X\times Q \xleftarrow{1_X\times t} X\times T \xrightarrow{\bar{p}} X$$

maps every $X\times\{q\}\times\{s\}$, where $s = \sigma_1\dots\sigma_n$, into X as $R_{\sigma_1}\circ\dots\circ R_{\sigma_n}$ whenever $(D_{\sigma_1}\circ\dots\circ D_{\sigma_n}(q))\cap T \neq \emptyset$ and as $\emptyset$ otherwise. Consequently, $(p^{\#})^{-1} \circ (1_X \times i)^{\#} \circ (1_X \times t)^{-1} \circ \bar{p}$ is equal to $|\mathcal{M}|$. Thus, by II.2,

$$\text{beh } \mathbb{M}(\mathcal{M}) = \alpha \circ |\mathcal{M}| \circ \omega .$$

Concluding remarks. In the present paper, we deal with F-machines only in the category Set. If K is a finitely complete category, $(\mathcal{E}, \mathcal{M})$ a factorization system in K, K is $\mathcal{M}$-well-powered and fulfils the $\mathcal{E}$-pullback property, then the category Rel K of relations in K can be formed and any $\mathcal{E}$-preserving functor $F:K \longrightarrow K$ extended to a mapping $\bar{F}$:Rel K$\rightarrow$ $\longrightarrow$ Rel K by the formula $F[\alpha,\beta] = [F(\alpha), F(\beta)]$ such that I.4.1)2)3) are fulfilled. This is presented in [T_1]. Then the notion of an F-machine, its run and behaviour can be formulated in this more general setting and the propositions I.9, I.10 and II.2 are still valid whenever $\mathcal{M}$-sub-objects of any object of K form a complete lattice.

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PURE SUBGROUPS SPLIT

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Abstract: The purpose of this note is to characterize a class of mixed abelian groups G having the property that each pure subgroup of G splits. For the groups of countable (torsionfree) rank the problem is solved completely.

Key words: Splitting group, generalized p-height, increasing p-height ordering, generalized p-sequence, p-rank.

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By the word "group" we shall always mean an additively written abelian group. If M is a subset of a group G, then $\langle M \rangle$ denotes the subgroup of G generated by M. If g is an element of infinite order of a mixed group G then $h_p^G(g)$ ($\tau^G(g)$) denotes the p-height (the characteristic) of g in the group G. The rank of a mixed group G with the maximal torsion subgroup T is the rank of the factor-group G/T.

In what follows we shall deal with a mixed group G with the maximal torsion subgroup T and $\overline{G}$ will denote the factor-group G/T. The bar over the elements will denote the elements from $\overline{G}$. We say that a set $M = \{a_\lambda \mid \lambda \in \Lambda\}$ of elements of G is a basis of G if the set $\overline{M} = \{\overline{a}_\lambda \mid \lambda \in \Lambda\}$ is a basis of $\overline{G}$, i.e. a maximal linearly independent subset of $\overline{G}$.

A sequence $g_0, g_1, \dots$ of elements of a mixed group G is said to be a p-sequence of g_0 if $pg_{i+1} = g_i$, $i = 0,1,\dots$. Let U be any torsionfree subgroup of a mixed group G and let $g \in G \setminus U$ be an element of infinite order. If $h_p^{G/U}(g+U) = \infty$ then every sequence $g = g_0, g_1, \dots$ of elements of G such that $p(g_{i+1}+U) = g_i+U$, $i = 0,1,\dots$, is called a generalized p-sequence of g with respect to U.

Let $M = \{a_\alpha \mid \alpha < \mu\}$ (μ is an ordinal number) be a well-ordered basis of a mixed group G. We define the generalized p-height $H_p^G(a_\alpha)$ of the element a_α as the p-height of $a_\alpha + \sum_{\beta<\alpha} \langle a_\beta \rangle$ in $G/\sum_{\beta<\alpha} \langle a_\beta \rangle$. The well-ordering on M is said to be an increasing p-height ordering if $H_p^G(a_\alpha) \le H_p^G(a_\beta)$ whenever $\alpha \le \beta < \mu$.

It is well-known (see [6]) that if H is a torsionfree group of finite rank and K its free subgroup of the same rank then the number $r_p(H)$ of summands $C(p^\infty)$ in H/K does not depend on the particular choice of K and this number is called the p-rank of H.

Lemma 1: Let $M = \{a_\lambda \mid \lambda \in \Lambda\}$ be a basis of a mixed group G with the torsion part T. Then G splits if and only if there are non-zero integers m_λ, $\lambda \in \Lambda$, such that

(1) $\tau^G(a) = \tau^{\bar{G}}(\bar{a})$ for each element $a \in \sum_{\lambda\in\Lambda} \langle m_\lambda a_\lambda \rangle$,

(2) for every prime p there is an increasing p-height ordering $\{m_\alpha a_\alpha \mid \alpha < \mu\}$ on $\widetilde{M} = \{m_\lambda a_\lambda \mid \lambda \in \Lambda\}$ such that $H_p^G(m_\alpha a_\alpha) = n_\alpha < \infty$ if and only if $\alpha < \nu$ and for every $\alpha < \nu$ there exists an element $x_\alpha \in G$ such that $p^{n_\alpha}(x_\alpha + \sum_{\beta<\alpha} \langle m_\beta a_\beta \rangle) = m_\alpha a_\alpha + \sum_{\beta<\alpha} \langle m_\beta a_\beta \rangle$ and every element $m_\gamma a_\gamma$, $\nu \le \gamma < \mu$, has a generalized p-sequence with respect to $U = \langle x_\alpha \mid \alpha < \nu \rangle$.

Proof: See [1; Theorem].

The definition of p-rank of a torsionfree group H (of arbitrary rank) can be found in [7]. In this note we shall need only the following result.

Lemma 2: If H is a torsionfree group, then $r_p(H) = 0$ if and only if $r_p(K) = 0$ for each pure subgroup K of H of finite rank.

Proof: See [8; Corollary 2].

Lemma 3: Let G be a mixed group with the torsion part T and p be a prime. Let $\{a_\alpha \mid \alpha < \mu\}$ be an increasingly p-height ordered basis of G such that $H_p^G(a_\alpha) = n_\alpha < \infty$ if and only if $\alpha < \nu$ and let $U = \langle x_\alpha \mid \alpha < \nu \rangle$ where $x_\alpha \in G$ are such that $p^{n_\alpha}(x_\alpha + \sum_{\beta<\alpha} \langle a_\beta \rangle) = a_\alpha + \sum_{\beta<\alpha} \langle a_\beta \rangle$. If the p-primary component T_p of T is a direct sum of a divisible and a bounded groups then every element a_γ, $\nu \leq \gamma < \mu$, has a generalized p-sequence with respect to U.

Proof: By hypothesis, $T_p = D \oplus V$ where D is divisible and $p^m V = 0$. Put $h_0 = a_\gamma$ and assume that we have constructed the elements $h_0, h_1, \ldots, h_n$ in such a way that h_0+U, $h_1+U, \ldots$ $\ldots, h_n+U$ are of infinite p-height in G/U and $p(h_{i+1}+U) = h_i+U$, $i = 0,1,\ldots,n-1$.

Since h_n+U is of infinite p-height in G/U, there exist elements $h^{(s)} \in G$, $u^{(s)} \in U$, $s = 1,2,\ldots$, such that $p^{m+s}h^{(s)} =$ $= h_n+u^{(s)}$. Then $p^{m+1}(p^{s-1}h^{(s)}-h^{(1)}) = u^{(s)}-u^{(1)}$ and $p^{m+1}w^{(s)} =$ $= u^{(s)}-u^{(1)}$ for some $w^{(s)} \in U$, U being p-pure in G by [1, Lemma 4]. Consequently, $p^{s+1}h^{(s)}-h^{(1)}-w^{(s)} = d^{(s)}+v^{(s)}$, $d^{(s)} \in D$, $v^{(s)} \in V$. From the divisibility of D the existence of elements $d_s^{(s)} \in D$ follows, for which $p^{s-1}d_s^{(s)} = d^{(s)}$. Now, putting

$h_{n+1} = p^m h^{(1)}$ and $z_s = h^{(s)} - d_s^{(s)}$, we have $ph_{n+1} = p^{m+1}h^{(1)} = h_n + u^{(1)}$, $p^{m+s-1}z_s = p^m(h^{(1)} + w^{(s)} + v^{(s)}) = h_{n+1} + p^m w^{(s)}$ and the assertion follows easily.

Lemma 4: Let S be a pure subgroup of a mixed group G with the torsion part T. Let p be a prime and $a \in S$ be an element of infinite order, $\tilde{S} = S/S \cap T$, $\tilde{a} = a + S \cap T$. If $h_p^{\bar{G}}(\bar{a}) = h_p^G(a)$ then $h_p^S(a) = h_p^{\tilde{S}}(\tilde{a}) = h_p^G(a)$.

Proof: Obviously, $h_p^S(a) = h_p^G(a) = h_p^{\bar{G}}(\bar{a}) \geqq h_p^{\tilde{S}}(\tilde{a}) \geqq h_p^S(a)$, as desired.

Lemma 5: Let G be a mixed group of the form $G = \sum_{i=1}^{\infty}{}^{\oplus} \langle t_i \rangle \oplus A = T \oplus A$ where $\langle t_i \rangle$ is a cyclic group of order p^{ℓ_i}, $\ell_1 < \ell_2 < \dots$, and A is a torsionfree group of finite rank. If $r_p(A) > 0$ then G contains a non-splitting pure subgroup.

Proof: We shall divide the proof into several steps.

a) If A contains a rank one p-divisible pure subgroup B then $T \oplus B$ is pure in G and $T \oplus B$ contains a non-splitting pure subgroup by [2; Lemma 12].

b) If $\{a_1, a_2, \dots, a_n, a_{n+1}\}$ is an increasingly p-height ordered basis of A then there is $k \leqq n$ such that $H_p^A(a_i) < \infty$ for each $i = 1, 2, \dots, k$ and $H_p^A(a_i) = \infty$ for each $i = k+1, \dots, n+1$. Obviously, we can assume that $k = n$, since in the opposite case we can treat the pure closure B of $\langle a_1, a_2, \dots, a_k, a_{k+1} \rangle$ in A instead of A.

c) In view of a), b) and [1; Lemma 4] we can suppose that A contains no element of infinite p-height and that it has a basis $\{a_1, a_2, \dots, a_n, a\}$ such that $\langle N \rangle = \langle a_1, a_2, \dots, a_n \rangle$ is p-pure in A and $h_p^{A/\langle N \rangle}(a + \langle N \rangle) = \infty$, $h_p^A(a) = 0$. Thus, there are elements $b_i \in A$ with $p^{\ell_i} b_i = a + v_i$, $v_i \in \langle N \rangle$, $i = 1,2,\ldots$. Put $s_i = b_i + t_i$, $i = 1,2,\ldots$, $U = \langle N \cup \{s_1, s_2, \ldots\} \rangle$ and $S = \{ s \in G \mid ms \in U$ for some integer m, $(m,p) = 1\}$. Obviously, S is π'-pure in G where $\pi' = \pi \setminus \{p\}$, π being the set of all primes.

d) Now we are going to show that S is pure in G. Suppose, at first, that the equation $p^k x = u$, $u \in U$, has the solution x in G. Let $x = \sum_{i=1}^{r} \mu_i t_i + a'$, $a' \in A$, and $u = v + \sum_{i=1}^{r} \lambda_i s_i$, $v \in \langle N \rangle$. Then $\sum_{i=1}^{r} p^k \mu_i t_i + p^k a' = v + \sum_{i=1}^{r} \lambda_i b_i + \sum_{i=1}^{r} \lambda_i t_i$, and so ($G$ splits) $\sum_{i=1}^{r} p^k \mu_i t_i = \sum_{i=1}^{r} \lambda_i t_i$, $p^k a' = v + \sum_{i=1}^{r} \lambda_i b_i$. Hence $\lambda_i = p^k \mu_i + p^{\ell_i} \nu_i$ for some integer ν_i, $i = 1,2,\ldots,r$. Let ℓ_j be such that $\ell_j \geq k$ and put $\nu = \sum_{i=1}^{r} \nu_i$, $u' = \sum_{i=1}^{r} \mu_i s_i + \nu p^{\ell_j - k} s_j$. Then $p^k u' = \sum_{i=1}^{r} \lambda_i s_i - \sum_{i=1}^{r} \nu_i (a + v_i) + \nu (a + v_j) = u - v - \sum_{i=1}^{r} \nu_i v_i + \nu v_j$. Further, $p^k(u' - x) = \nu v_j - v - \sum_{i=1}^{r} \nu_i v_i \in \langle N \rangle$ and $p^k v' = p^k(u' - x)$, $v' \in \langle N \rangle$, $\langle N \rangle$ being p-pure in A. So, $u = p^k x = p^k(u' - v')$ where $u' - v' \in U$.

Now the purity of S in G is easy to prove. If $p^k x = s$, $s \in S$, is solvable in G, then $ms = u \in U$ for some integer m, $(m,p) = 1$. So, there exist integers ϱ, σ with $m\varrho + p^k \sigma = 1$ and the preceding part yields the existence of $u' \in U$ such that $p^k u' = u$. Then $p^k(\varrho u' + \sigma s) = m\varrho s + p^k \sigma s = s$ and we are through.

e) Now we shall prove that $\langle t_j \rangle \cap S = 0$ for each $j = 1,2,\ldots$. If $p^k t_j \in S$ for some $k < \ell_j$ then there exists a positive integer m relatively prime to p such that $mp^k t_j = v + \sum_{i=1}^{r} \lambda_i s_i = v + \sum_{i=1}^{r} \lambda_i b_i + \sum_{i=1}^{r} \lambda_i t_i$, $v \in \langle N \rangle$. We can clearly assume that $r \geqq j$. The above equality yields $\lambda_i = p^{\ell_i}\mu_i$, $i = 1, 2,\dots,r$, $i \neq j$, $mp^k = \lambda_j - p^{\ell_j}\mu_j$ and $0 = p^{\ell_j - k}(v + \sum_{i=1}^{n} \lambda_i b_i) = p^{\ell_j-k}(v + \sum_{i=1}^{n} \mu_i(a+v_i) + mp^k b_j) = (p^{\ell_j-k}\sum_{i=1}^{n} \mu_i + m)a+w$, $w \in \langle N \rangle$. Hence $p^{\ell_j-k}\sum_{i=1}^{n} \mu_i + m = 0$, $p^{\ell_j-k} \mid m$ - a contradiction showing that $\langle t_j \rangle \cap S = 0$.

f) Suppose now that the group S splits, $S = P \oplus B$, P torsion, B torsionfree. Obviously, there exists a positive integer k such that $p^k a_1, p^k a_2,\dots,p^k a_n, p^k a \in B$. Put $\tilde{N} = \{p^k a_1, p^k a_2,\dots$ $\dots, p^k a_n\}$ and take an index j such that $\ell_j > k$. For each $i > j$ the equality $p^{\ell_i} b_i = a+v_i$ yields $p^{\ell_j}(p^{\ell_i-\ell_j} b_i - b_j) = v_i - v_j = p^{\ell_j} w_i$, $w_i \in \langle N \rangle$, $\langle N \rangle$ being p-pure in A. Further, for each $i > j$ the equality $p^{k+\ell_i} b_i = p^k a + p^k v_i$, $v_i \in \langle N \rangle$, yields $p^{k+\ell_i} c_i = p^k a + p^k v_i$, $c_i \in B$, B being pure in G. Hence $p^{\ell_j}(p^{\ell_i-\ell_j}p^k c_i - p^k c_j) = p^k(v_i - v_j) = p^{\ell_j}p^k w_i$ and so $p^{\ell_i-\ell_j}p^k c_i = p^k c_j + p^k w_i$, B being torsionfree, $p^k w_i \in \langle \tilde{N} \rangle \subseteqq B$. We have shown that $p^k c_j + \langle \tilde{N} \rangle$ is of infinite p-height in $G/\langle \tilde{N} \rangle$. Similarly, the element $p^k b_j + \langle \tilde{N} \rangle$ is of infinite p-height in $G/\langle \tilde{N} \rangle$ and the same property has the element $p^k b_j - p^k c_j + \langle \tilde{N} \rangle$. On the other hand, $p^{\ell_j}(p^k b_j - p^k c_j) = 0$ shows that $p^k b_j - p^k c_j + \langle \tilde{N} \rangle$ lies in the torsion part $T + \langle \tilde{N} \rangle / \langle \tilde{N} \rangle \cong T$ of $G/\langle \tilde{N} \rangle$ and so $p^k b_j = p^k c_j \in B$. Consequently, $p^k t_j = p^k s_j - p^k b_j \in S$ - a contradiction (see e)) finishing the proof.

Definition: We say that a torsionfree group G belongs to the class $\mathcal{W}$ if for each prime p with $r_p(G) = 0$ each linearly independent subset N of G can be increasingly p-height ordered in such a way that $N = \{a_\alpha \mid \alpha < \mu\}$ and $H_p^G(a_\alpha) < \infty$

for each $\alpha < \mu$.

<u>Theorem 1</u>: Let G be a mixed group with the torsion part T such that $\bar{G} \in \mathcal{W}$. Then every pure subgroup of G splits if and only if

(i) G contains a basis M such that $\tau^{G}(a) = \tau^{\bar{G}}(\bar{a})$ for each element $a \in \langle M \rangle$ and

(ii) T_p is a direct sum of a divisible and a bounded groups for each prime p with $r_p(\bar{G}) > 0$.

<u>Proof</u>: Sufficiency. Let p be a prime such that $r_p(\bar{G}) = 0$. Since $\bar{G} \in \mathcal{W}$, there exists an increasing p-height ordering $\{\bar{a}_\alpha, \alpha < \mu\}$ on the basis $\bar{M}$ of $\bar{G}$ such that $H_p^{\bar{G}}(\bar{a}_\alpha) < \infty$ for each $\alpha < \mu$. In view of (i), $H_p^{G}(a_\alpha) < \infty$ for each $\alpha < \mu$.

Let p be a prime with $r_p(\bar{G}) > 0$ and let $\{a_\alpha \mid \alpha < \mu\}$ be an increasing p-height ordering on M such that $H_p^{G}(a_\alpha) = n_\alpha < \infty$ if and only if $\alpha < \nu$. By Lemma 3, each element a_γ, $\nu \leq \gamma < \mu$, has a generalized p-sequence with respect to $U = \langle x_\alpha \mid \alpha < \nu \rangle$ where $x_\alpha \in G$ are such elements that $p^{n_\alpha}(x_\alpha + \sum_{\beta<\alpha} \langle a_\beta \rangle) = a_\alpha + \sum_{\beta<\alpha} \langle a_\beta \rangle$. Consequently, G splits by Lemma 1, $G = T \oplus A$.

Now let S be a pure subgroup of G and $N = \{a_\lambda \mid \lambda \in \Lambda\}$, be a basis of S. Then there exist non-zero integers m_λ, $\lambda \in \Lambda$, such that the basis $\tilde{N} = \{m_\lambda a_\lambda \mid \lambda \in \Lambda\}$ of S is contained in A. Hence $\tilde{N}$ satisfies condition (i) by Lemma 4.

If $r_p(\bar{G}) > 0$ then T_p is a direct sum of a divisible and a bounded groups by hypothesis. However, $(S \cap T)_p$ is pure in T_p by [2 ; Lemma 7] and $(S \cap T)_p$ is a direct sum of a divisible and a bounded groups by [2; Lemma 9].

Finally, suppose that $r_p(\overline{G}) = r_p(A) = 0$. Since $\overline{G} \in \mathcal{W}$ and $\widetilde{N}$ is a linearly independent subset of A, $\widetilde{N}$ can be increasingly p-height ordered in such a way that $\widetilde{N} = \{ m_\alpha a_\alpha | \alpha < \mu \}$ and $H_p^A(m_\alpha a_\alpha) = H_p^G(m_\alpha a_\alpha) = H_p^S(m_\alpha a_\alpha) < \infty$ for each $\alpha < \mu$.

Similar arguments as in the first part of the proof show that S splits.

Necessity. Condition (i) is necessary by Lemma 1. Assume that G does not satisfy the condition (ii). Thus for a prime p with $r_p(\overline{G}) > 0$ the p-primary component T_p is not a direct sum of a divisible and a bounded groups. Without loss of generality we can suppose that T_p is reduced and that $G = T' \oplus B$ splits. Then $r_p(B) = r_p(\overline{G}) > 0$ and Lemma 2 yields the existence of a pure subgroup A of B of finite rank with $r_p(A) > 0$. Each basic subgroup of T_p is unbounded by [2; Lemma 11] and so T_p contains a subgroup T pure in T' having the form $T = \sum_{i=1}^{\infty}{}^{\oplus} \langle t_i \rangle$ where $\langle t_i \rangle$ is a cyclic group of order ℓ_i, $\ell_1 < \ell_2 < \dots$. An application of Lemma 5 finishes the proof.

__Corollary 1__: Let $G = T \oplus A$, T torsion, A torsionfree, be a splitting group such that $A \in \mathcal{W}$. Then every pure subgroup of G splits if and only if T_p is a direct sum of a divisible and a bounded groups for each prime p with $r_p(A) > 0$.

__Proof__: Clearly, G satisfies condition (i) of Theorem 1 by Lemma 1.

__Lemma 6__: Every countable torsionfree group G belongs to the class $\mathcal{W}$.

__Proof__: Let p be such a prime that $r_p(G) = 0$ and let M be an arbitrary linearly independent subset of G. Choose $a_1 \in M$ in such a way that $h_p^G(a_1) = \min\{h_p^G(a) | a \in M\}$. It is obvious that $h_p^G(a_1) < \infty$ (since $r_p(G) = 0$). Suppose that we have constructed the elements $a_1, a_2, \dots, a_n$ such that $H_p^G(a_1) \leqq H_p^G(a_2) \leqq \dots \leqq H_p^G(a_n) \leqq H_p^G(a)$ for each $a \in M \setminus \{a_1, a_2, \dots, a_n\}$ and $H_p^G(a_n) < \infty$. Choose $a_{n+1} \in M \setminus \{a_1, a_2, \dots, a_n\}$ such that $h_p^{G/V}(a_{n+1}+V) = \min\{ h_p^{G/V}(a+V) \mid a \in M \setminus \{a_1, a_2, \dots, a_n\}\}$ where $V = \langle a_1, a_2, \dots, a_n \rangle$. Using Lemma 2 we see that $H_p^G(a_{n+1}) = h_p^{G/V}(a_{n+1}+V) < \infty$. Obviously, this procedure yields an increasing p-height ordering $\{a_1, a_2, \dots\}$ on M (M is countable by hypothesis) such that $H_p^G(a_i) < \infty$ for each $i = 1, 2, \dots$.

Theorem 2: Every pure subgroup of a mixed group G of countable (finite) rank splits if and only if

(i) G contains a basis M such that $\tau^G(a) = \tau^{\bar{G}}(\bar{a})$ for each element $a \in \langle M \rangle$ and

(ii) T_p is a direct sum of a divisible and a bounded groups for each prime p with $r_p(\bar{G}) > 0$.

Proof: It suffices to use Lemma 6 and Theorem 1.

Corollary 2: Let T be a torsion group and A be a countable torsionfree group. Then every pure subgroup of $G = T \oplus A$ splits if and only if T_p is a direct sum of a divisible and a bounded groups for each prime p with $r_p(A) > 0$.

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