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QUASIMODULES GENERATED BY THREE ELEMENTS

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Abstract: Quasimodules generated by three elements and their subquasimodules are investigated.

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This paper is a continuation of [1] and the reader is referred to [1] for definitions, basic properties of quasimodules, terminology, notation, references, etc.

1. Introduction. Throughout the paper, let R be a left noetherian associative ring with unit and $Z_3 = \{1,2,0\}$ the three-element field. Further, let $\Phi : R \longrightarrow Z_3$ be such that $-\Phi$ is a ring homomorphism of R onto Z_3. The word quasimodule will always mean a special left R-quasimodule of type (Φ).

For a set M, let $|M|$ designate the cardinal number corresponding to M. If Q is a quasimodule then o(Q) is the least cardinal number equal to $|M|$ for a generator set M of Q.

We shall define two primitive quasimodules $\underline{T}$ and $\underline{S}$ as follows:

$\underline{T} = Z_3$, + is the usual addition and $rx = -\Phi(r)x$.

$\underline{S} = \underline{S}(o,rx) = Z_3^4$, $\langle a,b,c,d\rangle \circ \langle x,y,u,v\rangle = \langle a+x, b+y, c+u, d+v+(ay-bx)(c-u)\rangle$ and $r\langle a,b,c,d\rangle = \langle -\Phi(r)a, -\Phi(r)b, -\Phi(r)c, -\Phi(r)d\rangle$.

1.1. Proposition. (i) $\underline{T}$ is a free primitive quasimodule of rank 1.

(ii) $\underline{T}^2$ is a free primitive quasimodule of rank 2.

(iii) $\underline{S}$ is a free primitive quasimodule of rank 3.

Proof. (i) and (ii). Every primitive quasimodule generated by at most two elements is a module. On the other hand, primitive modules are just vector spaces over Z_3.

(iii) One may verify easily that $\underline{S}$ is not a module and $\underline{S}$ is generated by three elements. Let Q be a free primitive quasimodule of rank 3. Q is generated by a set $\{a,b,c\}$ and Q is nilpotent of class at most 2 (see [1, Proposition 4.3]). Hence $K \subseteq A(Q) \subseteq C(Q)$ is a normal subquasimodule, where K is the subquasimodule generated by the associator (a,b,c). However, Q/K is a module by [1, Lemma 1.1] and consequently $K = A(Q)$, $o(A(Q)) \leq 1$ and $|A(Q)| \leq 3$, since $A(Q)$ is a primitive module. Finally, $o(Q/A(Q)) \leq 3$, $Q/A(Q)$ is a primitive module, $|Q/A(Q)| \leq 27$ and $|Q| \leq 81$. Since $\underline{S}$ is a homomorphic image of Q, Q is isomorphic to $\underline{S}$.

2. $\widetilde{\text{Soc}}$-torsion quasimodules

2.1. Lemma. Let Q be a quasimodule such that $o(Q/C(Q)) \leq 2$. Then Q is a module.

Proof. There are elements $a,b \in Q$ such that Q is generated by $C(Q) \cup \{a,b\}$. Denote by P the subquasimodule generated by these elements. Then P is a module and Q is a homomorphic image of the product $C(Q)\times P$. Hence Q is a module.

2.2. <u>Lemma</u>. Let Q be a primitive module and $0\le n$. Then $o(Q) = n$ iff Q is finite and $|Q| = 3^n$.

Proof. The variety of primitive modules is equivalent to the variety of abelian groups with $3x = 0$. The rest is clear.

2.3. <u>Lemma</u>. Let Q be a finitely generated primitive quasimodule. Then Q is finite and $|Q| = 3^n$ for some $0\le n$.

Proof. Q is nilpotent and we can proceed by the nilpotent class m of Q. If $m\le 1$ then Q is a module and the result follows from 2.2. Let $2\le m$. Then $Q/C(Q)$ is nilpotent of class at most m-1 and $C(Q)$ is a finitely generated primitive module. The rest is clear.

2.4. <u>Proposition</u>. Let Q be a finitely generated $\tilde{\mathcal{K}}$-torsion quasimodule. Then Q is finite and $|Q| = 3^n$ for some $0\le n$.

Proof. Q is noetherian and $\tilde{\mathcal{K}}$-torsion. Hence there is a finite sequence $0 = P_0 \subseteq P_1 \subseteq \dots \subseteq P_{m-1} \subseteq P_m = Q$ of normal subquasimodules such that P_i/P_{i-1} are finitely generated and primitive. It remains to apply 2.3.

2.5. <u>Proposition</u>. Suppose that the ring R has primary decompositions. Let $\mathcal{A}$ be a representative set of simple modules and Q a $\widetilde{\mathrm{Soc}}$-torsion quasimodule. Then Q is a direct sum of its subquasimodules $\widetilde{\mathrm{Soc}}_S(Q)$, $S\in\mathcal{A}$.

Proof. It suffices to show that Q is generated by $\bigcup \widetilde{\mathrm{Soc}}_S(Q)$. However, this is clear from the fact that $A(Q)\subseteq \subseteq \mathcal{K}(Q)$.

2.6. <u>Proposition</u>. Suppose that the ring R has primary decompositions. Let Q be a finitely generated $\widetilde{\mathrm{Soc}}$-torsion quasimodule. Then there is a finite set $S_1,\dots,S_n$, $0\le n$, of simple modules not isomorphic to $\mathcal{I}$ such that Q is isomorphic to the product $\widetilde{\mathcal{K}}(Q)\times \mathrm{Soc}_{S_1}(Q)\times\dots\times\widetilde{\mathrm{Soc}}_{S_n}(Q)$. Moreover, if Q is not a module then $\widetilde{\mathcal{K}}(Q)\neq 0$.

Proof. Apply 2.5 and [1, Lemma 4.16].

2.7. <u>Proposition</u>. Suppose that R is commutative and finitely generated. Then every finitely generated $\widetilde{\mathrm{Soc}}$-torsion quasimodule is finite.

Proof. This is an easy consequence of 2.4 and 2.6 (take into account that every simple module is finite).

2.8. <u>Proposition</u>. Suppose that R is commutative and finitely generated. Then every finite directly indecomposable quasimodule is either a module or $\widetilde{\mathcal{K}}$-torsion.

Proof. Apply 2.6.

2.9. <u>Lemma</u>. Let $1\le n$ and Q be a quasimodule which is not nilpotent of class at most n. Then $3^{2n+2}\le |Q|$.

Proof. We can assume that Q is finite and subdirectly irreducible. Then Q is nilpotent of class m, $n+1\le m$. In particular, $n+2\le o(Q)$. But $A(Q)\subseteq \mathcal{J}(Q)$, and so $n+2\le o(Q/A(Q))$ (use [1, Proposition 4.12]). On the other hand, Q and $Q/A(Q)$ are $\widetilde{\mathcal{K}}$-torsion. Hence $3^{n+2}\le |Q/A(Q)|$. Finally, $0\subsetneqq A_n(Q)\subsetneqq\dots$ $\dots\subsetneqq A_2(Q)\subsetneqq A(Q)\subsetneqq Q$. Thus $3^n\le |A(Q)|$ and $3^{2n+2}\le |Q|$.

2.10. <u>Corollary</u>. Let Q be a non-associative quasimodule. Then $81\le |Q|$.

3. <u>The radical E</u>. Put $E = p_{\mathcal{P}}$. That is, for a quasimodule Q, E(Q) is just the least normal subquasimodule such that

the corresponding factor is primitive.

3.1. Lemma. Let Q be a quasimodule. Then E(Q) is just the subloop generated by the elements $rx + \Phi r(x)$, $x \in Q, r \in R$.

Proof. Denote by P the subloop. Obviously, P is a subquasimodule (we have $srx + s \Phi (r)x = (s \Phi (r)x + \Phi (s) \Phi (r)x) + (srx + \Phi (sr)x))$ and $P \subseteq E(Q)$. On the other hand, $P \subseteq C(Q)$, P is normal and Q/P is primitive. Thus P = E(Q).

3.2. Lemma. Suppose that the ring R and a quasimodule Q are generated by subsets M and N, resp. Denote by P the subquasimodule generated by the elements $rx + \Phi (r)x$, $r \in M$, $x \in N$. Then P = E(Q).

Proof. It is easy to see that $rx + \Phi (r)x \in P$ for all $x \in Q$ and $r \in M$. Denote by K the set of all $r \in R$ such that $rx + \Phi (r)x \in P$ for every $x \in Q$. We have $M \subseteq K$ and K(+) is a subgroup of R(+). Let $r, s \in K$ and $x \in Q$. Then $rsx + \Phi (rs)x = rsx - \Phi (r) \Phi (s)x = (rsx + r \Phi (s)x) + (-r \Phi (s)x - \Phi (r) \Phi (s)x) \in P$. Thus K is a subring of R and K = R.

3.3. Proposition. E is a cohereditary radical for $\mathscr{S}$. Moreover, $D \subseteq E \subseteq C$ and $\mathfrak{J} \subseteq A + E$.

Proof. Easy (use 3.1).

3.4. Proposition. Suppose that $R = Z[\alpha_1, \dots, \alpha_n]$, $0 \leq n$, is the ring of polynomials with n commuting indeterminates over the ring Z of integers. Then $A(Q) \cap E(Q) = 0$ for every free quasimodule Q.

Proof. We shall proceed by induction on n. First, let n = 0. Then, by 3.2, E(Q) = D(Q) = 3Q. Let $a \in A(Q) \cap E(Q)$ and let f denote the natural homomorphism of Q onto Q/A(Q). We have a = 3b for some $b \in Q$, so 3f(b) = 0. But Q/A(Q) is a free

Z-module, i.e. an abelian group, and therefore $f(b) = 0$, $b \in A(Q)$. Since $A(Q)$ is primitive, $a = 3b = 0$. Now, let $1 \le n$. Denote by P the subquasimodule generated by $\alpha_1 x + \Phi(\alpha_1)x$, $x \in Q$. Since $P \subseteq C(Q)$, P is a normal submodule. Moreover, $P = \{(\alpha_1 + \Phi(\alpha_1))x \mid x \in Q\}$. Let $G = Q/P$ and let g denote the natural homomorphism of Q onto G. First, we show that $A(Q) \cap \cap P = 0$. For, let $a \in A(Q) \cap P$. We have $a = (\alpha_1 + \Phi(\alpha_1))b$ for some $b \in Q$, $(\alpha_1 + \Phi(\alpha_1))f(b) = 0$ in $Q/A(Q)$ and $f(b) = 0$. Thus $b \in A(Q)$ and $a = 0$, $A(Q)$ being primitive. Now, the quasimodule G can be considered a $Z[\alpha_2, \dots, \alpha_n]$-quasimodule (we have $\alpha_1 x = -\Phi(\alpha_1)x$ for every $x \in Q$). In this case, it is free and $A(G) \cap E(G) = 0$ by the induction hypothesis. Let $a \in A(Q) \cap E(Q)$. Then $g(a) \in A(G) \cap E(G)$, $g(a) = 0$, $a \in A(Q) \cap E(Q) \cap \cap P = 0$.

3.5. <u>Proposition</u>. Suppose that R is commutative and finitely generated. Then $A(Q) \cap E(Q) = 0$ for every free quasimodule Q.

Proof. There are a polynomial ring $P = Z[\alpha_1, \dots, \alpha_n]$ and a surjective ring homomorphism $\varphi : P \to R$ preserving the unit. Put $\Psi = \Phi\varphi$ and let Q be a free R-quasimodule. Then there are a free P-quasimodule F of type (Ψ) and a homomorphism f of F onto Q. Let $x \in A(Q) \cap E(Q)$. There are $a \in A(F)$ and $b \in E(F)$ with $f(a) = x = f(b)$. Then $a - b \in \operatorname{Ker} f$. But $\operatorname{Ker} f = IF$, where $I = \operatorname{Ker} \varphi$. Since $I \subseteq \operatorname{Ker}(-\Phi\varphi)$, $\operatorname{Ker} f \subseteq E(F)$ and $a \in A(F) \cap E(F) = 0$. Thus $a = 0$ and $x = 0$.

3.6. <u>Lemma</u>. Let P be a normal subquasimodule of a quasimodule Q such that $P \cap A(Q) = 0$. Then $P \subseteq C(Q)$.

Proof. For $x \in P$, $a, b \in Q$, $((x+a)+b)-(x+(a+b)) \in P \cap A(Q)$.

Hence $(x+a)+b = x+(a+b)$ and $x \quad C(Q)$.

3.7. Lemma. Let P be a normal subquasimodule of a quasimodule Q such that $P \cap E(Q) = 0$. Then $P \subseteq \mathcal{K}(Q)$.

Proof. Obvious.

4. Quasimodules generated by three elements. Throughout this section, let Q be a non-associative quasimodule with $o(Q) = 3$.

4.1. Proposition. (i) Q is nilpotent of class 2.

(ii) $A(Q) \subseteq C(Q)$ and $A(Q)$ is isomorphic to $\underline{T}$.

(iii) $Q/C(Q)$ is isomorphic to $\underline{T}^3$.

(iv) Either $E(Q) = C(Q)$ or $Q/E(Q)$ is isomorphic to $\underline{S}$ and $C(Q)/E(Q)$ to $\underline{T}$.

(v) If $E(Q) \neq C(Q)$ then $E(Q) \cap A(Q) = 0$.

(vi) $C(Q) = A(Q) + E(Q)$.

Proof. (i) This is clear.

(ii) By (i), $A(Q) \subseteq C(Q)$. Further, there are $a,b,c \in Q$ such that Q is generated by these elements. Let P be the subloop of $Q(+)$ generated by $((a+b)+c) - (a+(b+c))$. Then $P \subseteq A(Q) \subseteq C(Q)$, and hence P is a normal submodule of Q. By [1, Lemma 4.5], Q/P is a module. Hence $P = A(Q)$ and $o(A(Q)) \leq 1$. However, $A(Q) \neq 0$ is a primitive module. Consequently $A(Q)$ is isomorphic to $\underline{T}$.

(iii) By 2.1, $o(Q/C(Q)) = 3$. However, $Q/C(Q)$ is a primitive module and consequently $Q/C(Q)$ is isomorphic to $\underline{T}^3$.

(iv) and (v). Let $E(Q) \neq C(Q)$ and $P = Q/E(Q)$. Then P is primitive and P is a homomorphic image of $\underline{S}$. On the other hand, $27 = |Q/C(Q)| \leq |P|$, $|P| = 81 = |\underline{S}|$, and P is isomorphic to $\underline{S}$. In particular, P is not a module, $A(Q) \nsubseteq E(Q)$ and $A(Q) \cap E(Q) = 0$, since $A(Q)$ is simple.

(vi) Put $P = A(Q) + E(Q)$. We have $P \subseteq C(Q)$ and Q/P is a primitive module generated by three elements. Thus $27 = |Q/P|$ and $P = C(Q)$.

4.2. Lemma. Let P be a proper subquasimodule of Q such that $C(Q)$ is contained in P. Then P is a module.

Proof. Obviously, $f(P)$ is a proper subquasimodule of $Q/C(Q)$, where $f: Q \longrightarrow Q/C(Q)$ is the natural homomorphism. By 4.1(iii), $o(f(P)) \leq 2$. But $C(Q) \subseteq C(P)$, hence $o(P/C(P)) \leq 2$ and P is a module by 2.1.

4.3. Lemma. Let P be a maximal submodule of Q. Then P is a normal maximal subquasimodule and Q/P is isomorphic to $\underline{T}$. Moreover, $C(Q)$ is contained in P.

Proof. The set $C(Q) + P$ is a submodule of Q. Hence $C(Q) \subseteq P$ and P is a normal maximal subquasimodule of Q by 4.2. Finally, Q/P is simple and a homomorphic image of $Q/C(Q)$. Thus Q/P is isomorphic to $\underline{T}$.

4.4. Lemma. Let P be a submodule of Q. Then $E(Q) + P \neq Q$.

Proof. There is a maximal submodule G of Q such that $P \subseteq G$. By 4.3, $E(Q) + P \subseteq C(Q) + P \subseteq G$.

4.5. Lemma. Let P be a normal subquasimodule of Q such that $A(Q) \nsubseteq P$. Then P is a module and $P \subseteq C(Q)$. Moreover, if $\underline{S}$ is a homomorphic image of Q/P then $P \subseteq E(Q)$.

Proof. Since $A(Q) \nsubseteq P$, $P \cap A(Q) = 0$. By 3.6, $P \subseteq C(Q)$. The rest is clear.

4.6. Proposition. A subquasimodule P of Q is normal iff either $A(Q) \subseteq P$ or $P \subseteq C(Q)$.

Proof. First, let P be normal. If $A(Q) \nsubseteq P$ then $P \subseteq C(Q)$

by 4.5. Conversely, if $A(Q) \subseteq P$ then P is normal, since $Q/A(Q)$ is a module. The other case is clear.

4.7. <u>Corollary</u>. Let P be a normal subquasimodule of Q. Then either P or Q/P is a module.

4.8. <u>Lemma</u>. Let P be a subquasimodule of Q such that P is not a module. Then $A(Q) \subseteq P$, P is normal, $E(Q) + P = Q$ and $\underline{T}$ is not a homomorphic image of Q/P.

Proof. We have $0 \neq A(P) \subseteq A(Q)$. Hence $A(P) = A(Q)$ and P is normal. Further, suppose that Q/K is isomorphic to $\underline{T}$ for a normal subquasimodule K with $P \subseteq K$. Then $A(Q), E(Q) \subseteq K$, $C(Q) = A(Q) + E(Q) \subseteq K$ and K is a module by 4.2, a contradiction. Now, it is clear that $E(Q) + P = Q$.

4.9. <u>Lemma</u>. $\underline{S}$ is a homomorphic image of Q iff $E(Q) \neq C(Q)$.

Proof. If $E(Q) \neq C(Q)$ then $Q/E(Q)$ is isomorphic to $\underline{S}$ by 4.1(iv). Let $\underline{S}$ be a homomorphic image of Q. Then $Q/E(Q)$ is not a module, and so $E(Q) \neq C(Q)$.

4.10. <u>Proposition</u>. $E(Q) \neq C(Q)$ iff Q is a subdirect product of $\underline{S}$ and a module.

Proof. Apply 4.1(iv),(v) and 4.9.

4.11. <u>Construction</u>. Suppose that $E(Q) \neq C(Q)$. Then $A(Q) \cap E(Q) = 0$. Denote by f and g the natural homomorphisms of Q onto $Q/A(Q)$ and Q onto $Q/E(Q)$, resp. By 4.1(iv), $Q/E(Q)$ is isomorphic to $\underline{S}$. Moreover, $g(C(Q)) \subseteq C(Q/E(Q))$ and $0 \neq g(C(Q))$. Hence $g(C(Q)) = C(Q/E(Q))$ is isomorphic to $\underline{T}$ and $g(C(Q)) = \{0,x,y\}$. Let $a,b \in C(Q)$ be such that $g(a) = x$ and $g(b) = y$. Then $C(Q)$ is the disjoint union of the sets $E(Q)$, $a+E(Q)$, $b+E(Q)$. Since $C(Q) = A(Q) + E(Q)$, $f(E(Q)) = f(a+E(Q)) = f(b+E(Q)) = f(C(Q))$. Consider a subquasimodule G of $f(C(Q))$ and a homomorphism h of

G onto $g(C(Q))$. Then $G/\mathrm{Ker}\ h$ is isomorphic to $\underline{T}$ and h induces an isomorphism k from $G/\mathrm{Ker}\ h$ onto $g(C(Q))$. Finally, let $p: f(C(Q)) \to f(C(Q))/G$ and $q: f(C(Q)) \to f(C(Q))/\mathrm{Ker}\ h$ be the natural homomorphisms. Denote by P the set of all $c \in C(Q)$ with $f(c) \in G$ and $hf(c) = g(c)$.

4.11.1. Lemma. P is a submodule of $C(Q)$, $A(Q) \cap P = 0$ and $P \nsubseteq E(Q)$.

Proof. Obviously, P is a submodule of $C(Q)$. Let $c \in A(Q) \cap$ $\cap P$. Then $g(c) = hf(c) = 0$, $c \in E(Q) \cap A(Q) = 0$. Further, let $z \in G$ be such that $h(z) = x$. As $f(a+E(Q)) = f(C(Q))$, $z = f(a+c)$ for some $c \in E(Q)$. We have $f(a+c) = z \in G$ and $hf(a+c) = h(z) =$ $= x = g(a+c)$. Hence $a+c \in P$. But $g(a+c) = x \neq 0$, and so $a+c \notin$ $\notin E(Q)$.

4.11.2. Lemma. P is a normal submodule of Q, $A(Q) \nsubseteq P$ and $\underline{S}$ is not a homomorphic image of Q/P.

Proof. P is normal, since it is contained in $C(Q)$. Further, $A(Q) \nsubseteq P$ by 4.11.1 and $\underline{S}$ is not a homomorphic image of Q/P due to 4.11.1 and 4.5.

4.11.3. Lemma. $C(Q)/P$ is isomorphic to $f(C(Q))/\mathrm{Ker}\ h$.

Proof. Define a mapping t of $C(Q)$ into $f(C(Q))/\mathrm{Ker}\ h$ by $t(c) = k^{-1}g(c) - qf(c)$ for every $c \in C(Q)$. Using the fact that $f(C(Q))/\mathrm{Ker}\ h$ is a module, it is easy to see that t is a homomorphism. If $c \in P$ then $t(c) = k^{-1}hf(c) - qf(c) = 0$, and so $P \subseteq \mathrm{Ker}\ t$. Conversely, if $c \in \mathrm{Ker}\ t$, then $k^{-1}g(c) = qf(c)$, $f(c) \in$ $\in G$ and $g(c) = hf(c)$, $c \in P$. Thus $\mathrm{Ker}\ t = P$ and it remains to show that $t(C(Q)) = f(C(Q))/\mathrm{Ker}\ h$. For, let $z \in f(C(Q))/\mathrm{Ker}\ h$ be an element. We have $z = qf(c)$ for some $c \in E(Q)$ and $t(-c) =$ $= qf(c) - k^{-1}g(c) = qf(c) = z$.

4.12. <u>Lemma</u>. Suppose that $E(Q) \neq C(Q)$. Let P be a normal subquasimodule of Q such that $A(Q) \nsubseteq P$ and $\underline{S}$ is not a homomorphic image of Q/P. Then P is a submodule of the type constructed in 4.11.

Proof. By 4.1 and 4.5, $P \subseteq C(Q)$ and $P \nsubseteq E(Q)$. Let $f:Q \longrightarrow Q/A(Q)$ and $g:Q \longrightarrow Q/E(Q)$ be the natural homomorphisms. As we know, $g(C(Q)) = \{0,x,y\}$ is isomorphic to $\underline{T}$. Since $P \nsubseteq E(Q)$, $g(P) = g(C(Q))$. Furthermore, $A(Q) \cap P = 0$ and $f|P:P \longrightarrow f(P)$ is an isomorphism. Consequently there is a homomorphism $h:f(P) \longrightarrow g(P)$ such that $hf(c) = g(c)$ for every $c \in P$. Obviously, $hf(P) = g(C(Q))$. Put $f(P) + G$. If $c \in P$ then $f(c) \in G$ and $hf(c) = g(c)$. Conversely, if $c \in C(Q)$, $f(c) \in G$ and $hf(c) = g(c)$, then $f(c) = f(d)$ for some $d \in P$ and we can write $g(c) = hf(c) = hf(d) = g(d)$. Thus $c - d \in A(Q)\ E(Q) = 0$, $c = d$ and $c \in P$. The rest is clear.

4.13. <u>Theorem</u>. Let Q be a non-associative quasimodule with $o(Q) = 3$. Let P be a subquasimodule of Q. Then:

(i) P is normal, Q/P is a module and $\underline{T}$ is not a homomorphic image of Q/P iff P is not a module.

(ii) P is normal, Q/P is a module and $\underline{T}$ is a homomorphic image of Q/P iff P is a module and $A(Q) \subseteq P$.

(iii) P is normal, Q/P is not a module and $\underline{S}$ is not a homomorphic image of Q/P iff $P \subseteq C(Q)$ and either $E(Q) = C(Q)$ and $P \cap A(Q) = 0$ or $E(Q) \neq C(Q)$ and P is a submodule of the type constructed in 4.11.

(iv) P is normal and $\underline{S}$ is a homomorphic image of Q/P iff $E(Q) \neq C(Q)$ and $P \subseteq E(Q)$.

Proof. Apply the preceding results.

4.14. <u>Lemma</u>. Let f be a homomorphism of a quasimodule Q onto a quasimodule P. Suppose that P is not a module and $o(Q) \leq 3$. Then $f(C(Q)) = C(P)$.

Proof. By 4.5, $\mathrm{Ker}\, f \subseteq C(Q)$ and $P/f(C(Q))$ is isomorphic to $Q/C(Q)$. According to 4.1, $P/C(P)$ is isomorphic to $Q/C(Q)$. Now, it is obvious that $C(P) = f(C(Q))$.

5. <u>Several consequences</u>. In this section, suppose that R is commutative.

5.1. <u>Proposition</u>. Let Q be a $\mathfrak{X}$-torsion quasimodule such that $o(Q) \leq 3$. Then every proper subquasimodule of Q is a module.

Proof. We can assume that Q is not a module. Let P be a proper subquasimodule such that P is not a module. Since Q is noetherian, we can assume that Q is a maximal subquasimodule. By 4.8, P is normal and Q/P is not isomorphic to $\underline{T}$, a contradiction.

5.2. <u>Proposition</u>. Let Q be a subdirectly irreducible quasimodule nilpotent of class 2. Then Q is $\mathfrak{X}$-torsion and $A(Q) \neq 0$ is the least non-zero normal subquasimodule of Q. Moreover, $A(Q)$ is isomorphic to $\underline{T}$ and every proper factorquasimodule of Q is a module.

Proof. Since Q is nilpotent of class 2, $0 \neq A(Q) \subseteq C(Q)$. By [1, Proposition 5.4], Q is $\mathfrak{X}$-torsion. Further, $A(Q)$ is a subdirectly irreducible primitive module. Hence $A(Q)$ is isomorphic to $\underline{T}$ and the rest is evident.

We shall say that a quasimodule Q satisfies the condition (α) if Q is not a module and every proper subquasimodule as well as factorquasimodule of Q is a module.

5.3. Theorem. The following conditions are equivalent for a non-associative quasimodule Q:

(i) Q satisfies (α).

(ii) Every subquasimodule and every factorquasimodule of Q is either a module or isomorphic to Q.

(iii) Q is subdirectly irreducible and every subquasimodule of Q is either a module or isomorphic to Q.

(iv) Q is subdirectly irreducible and $o(Q) \leq 3$.

(v) $o(Q) \leq 3$ and every factorquasimodule of Q ie either a module or isomorphic to Q.

Proof. (i) implies (ii). This is trivial.

(ii) implies (iii). Q is not a module, and hence there is a subdirectly irreducible factor P of Q such that P is not a module. Thus P is isomorphic to Q.

(iii) implies (iv). There are $a,b,c \in Q$ with $a+(b+c) \neq (a+b)+c$. Denote by P the subquasimodule generated by these elements. Then P is not associative and P is isomorphic to Q.

(iv) implies (v) and (i). Apply 5.1 and 5.2.

(v) implies (iv). This is easy.

5.4. Proposition. Let Q be a quasimodule satisfying (α). Then:

(i) Q is subdirectly irreducible, nilpotent of class 2 and $o(Q) = 3$.

(ii)$^{\bullet}$ Q is $\mathfrak{X}$-torsion, finite and $|Q| = 3^n$ for some $4 \leq n$.

(iii) $0 \neq A(Q) \subseteq \mathcal{J}(Q) = C(Q) = A(Q) + E(Q)$ and $A(Q) = C(Q) \cap \mathfrak{X}(Q)$.

(iv) $A(Q)$ is isomorphic to $\underline{T}$ and $Q/C(Q)$ to $\underline{T}^3$.

(v) Q is isomorphic to $\underline{S}$, provided Q is primitive.

(vi) If Q is not primitive then $\mathcal{J}(Q) = E(Q) = C(Q)$.

Proof. (i) See 5.3.

(ii) Use 5.2, 2.4 and 2.10.

(iii) Since Q is not associative, $0 \neq A(Q)$, Moreover, $A(Q) \subseteq \mathcal{J}(Q)$ by [1, Lemma 4.20] and $C(Q) = A(Q) + E(Q)$ by 4.1 (vi). On the other hand, every simple factor of Q is isomorphic to $\underline{T}$, and so $E(Q) \subseteq \mathcal{J}(Q)$. In particular, $C(Q) = A(Q) + E(Q) \subseteq \mathcal{J}(Q)$. However, by [1, Proposition 4.12], $o(Q/\mathcal{J}(Q)) = 3$, hence $|Q/\mathcal{J}(Q))| = |Q/C(Q))|$ and $\mathcal{J}(Q) = C(Q)$. Finally, $C(Q) \cap \mathcal{K}(Q)$ is a subdirectly irreducible primitive module. The rest is clear.

(iv) Apply 5.2 and 4.1.

(v) Let Q be primitive. Then Q is a homomorphic image of $\underline{S}$. Thus Q is isomorphic to $\underline{S}$.

(vi) Let Q be not primitive. Then $E(Q) \neq 0$, $A(Q) \subseteq E(Q)$ and $E(Q) = C(Q)$.

5.5. Proposition. A quasimodule Q is not associative iff there are two subquasimodules G, H of Q such that G is a normal subquasimodule of H and H/G is a quasimodule satisfying (α).

Proof. It suffices to show the direct implication. Since Q is not a module, $a+(b+c) \neq (a+b)+c$ for some $a,b,c \in Q$. Let H be the subquasimodule generated by these elements. Then H is not associative and there is a normal subquasimodule G of H such that H/G is subdirectly irreducible and not associative. By 5.3, H/G satisfies (α).

5.6. Theorem. Let R be a principal ideal domain. Then, for every $4 \leq n$, there exists a quasimodule Q such that Q satisfies (α), $|Q| = 3^n$ and Q is not primitive.

Proof. Let F be a free quasimodule of rank three and let f denote the natural homomorphism of F onto $F/A(F)$, By 4.1, $0 = A(F) \cap E(F)$, $0 \neq E(F)$ and $C(F) = A(F) + E(F)$. In particular, $0 \neq f(C(F))$ is a free module. Hence, there are two submodules G, H of $F(C(F))$ such that $H \subseteq G$, G/H is isomorphic to $\underline{T}$ and $f(C(F))/H$ is a $\mathfrak{X}$-torsion subdirectly irreducible cyclic module with 3^{n-3} elements. Further, let $g: F \longrightarrow F/E(F)$ be the natural homomorphism. Then $g(C(F)) = C(F/E(F))$ is isomorphic to $\underline{T}$ (use 4.14). Hence there is a homomorphism h of G onto $g(C(F))$ such that $H = \mathrm{Ker}\, h$. Consider the submodule P of $C(F)$ corresponding to G, h in the sense of 4.11 and put $Q =$ $= F/P$. By 4.11.2, Q is not associative and $\underline{S}$ is not a homomorphic image of Q. We have $o(Q) = 3$. By 4.14 and 4.11.3, $C(Q) =$ $= C(F)/P$ is isomorphic to $f(C(F))/H$. In particular, $C(Q)$ is subdirectly irreducible and Q is subdirectly irreducible by [1, Proposition 5.3]. By 5.3, Q satisfies (α). Furthermore, $|C(Q)| = 3^{n-3}$ and $|Q/C(Q)| = 27$. Thus $|Q| = 3^n$. Finally, Q is not primitive, since $\underline{S}$ is not a homomorphic image of Q.

6. Free quasimodules

6.1. **Lemma**. Let $0 \neq n$ and Q be a quasimodule such that $o(Q) \leq n$ and $Q/A(Q)$ is a free module of rank n. Suppose that $|A(P)| \leq |A(Q)|$, where P is a free quasimodule of rank n. Then Q is isomorphic to P.

Proof. Since $o(Q) \leq n$, there is a homomorphism f of P onto Q. Further, let $g: P \longrightarrow P/A(P)$ and $k: Q \longrightarrow Q/A(Q)$ be the natural homomorphisms. Since $f(A(P)) = A(Q)$, f induces a homomorphism h of $P/A(P)$ onto $Q/A(Q)$. However, both $P/A(P)$ and $Q/A(Q)$ are free modules of the same finite rank and consequently h is an isomorphism. Now, let $a \in P$ and $f(a) = 0$. Then $hg(a) =$

$= kf(a) = 0$, $g(a) = 0$, $a \in A(P)$. Thus $\mathrm{Ker}\, f \subseteq A(P)$. On the other hand, $|A(P)| \leq |A(Q)|$ and $f(A(P)) = A(Q)$. Since $A(Q)$ is finite, $f|A(P)$ is injective and $\mathrm{Ker}\, f = 0$.

6.2. Proposition. Let Q be a quasimodule and P be a free quasimodule of a finite rank n. Suppose that $o(Q) \leq n$ and P is a homomorphic image of Q. Then Q is isomorphic to P.

Proof. Put $G = Q/A(Q)$. Then $o(G) \leq n$ and $P/A(P)$ is a homomorphic image of G. But $P/A(P)$ is a free module of rank n. Hence $P/A(P)$ is isomorphic to G. The rest follows from 6.1.

In the remaining part of the paper, assume that R is a principal ideal domain.

6.3. Proposition. Let Q be a free quasimodule and P be a submodule of Q. Then there are a free module G and a primitive quasimodule H such that P is isomorphic to $G \times H$.

Proof. Denote by f the natural homomorphism of Q onto $Q/A(Q)$. Then $f(P)$ is a free module and consequently P is isomorphic to the product $f(P) \times H$, where $H = \mathrm{Ker}\, f \cap A(Q)$.

6.4. Lemma. Let Q be a finitely generated quasimodule such that Q is not associative, $o(Q/A(Q)) \leq 3$ and $\mathrm{Soc}(Q/A(Q)) = 0$. Then Q is free of rank 3.

Proof. Since $A(Q) \subseteq \mathcal{J}(Q)$, $o(Q/\mathcal{J}(Q)) = o(Q)$ and Q is not associative, $o(Q) = p(Q/A(Q)) = 3$. On the other hand, $Q/A(Q)$ is a finitely generated module with zero socle. Therefore $Q/A(Q)$ is a free module. Finally, let P be a free quasimodule of rank 3. Then $A(P)$ is isomorphic to $\underline{T}$, and so it is a homomorphic image of $A(Q)$. By 6.1, Q is isomorphic to P.

6.5. Proposition. Let Q be a free quasimodule of rank 3. Then $A(Q) = \mathcal{K}(Q)$ is isomorphic to $\underline{T}$, $E(Q)$ to R^3 and $C(Q)$ to

$R^3 \times \underline{T}$. Hence $o(C(Q)) = 4$.

Proof. $A(Q) = \mathcal{K}(Q)$, since $\mathcal{K}(Q/A(Q)) = 0$. By 4.1, $A(Q)$ is isomorphic to $\underline{T}$. Further, $Q/E(Q)$ is isomorphic to $\underline{S}$, $E(Q) \cap A(Q) = 0$ and $C(Q) = E(Q) + A(Q)$. Thus $C(Q)$ is isomorphic to $E(Q) \times \underline{T}$ and $E(Q)$ to $E(Q/A(Q))$. However, $E(Q/A(Q))$ is isomorphic to $E(R^3)$ and $E(R^3)$ is isomorphic to R^3.

6.6. <u>Theorem</u>. Let Q be a free quasimodule of rank 3. A quasimodule P is isomorphic to a subquasimodule of Q iff it is isomorphic to one of the following quasimodules: 0, $\underline{T}$, R, R^2, R^3, $R \times \underline{T}$, $R^2 \times \underline{T}$, $R^3 \times \underline{T}$, Q. Hence P is isomorphic to Q, provided it is not a module.

Proof. First, let P be a subquasimodule of Q. The factor $Q/A(Q)$ is a free module of rank 3. If P is not associative then $A(P) = A(Q)$ and $P/A(P)$ is a free module. By 6.4, P is isomorphic to Q. Now, suppose that P is a module. In this case, we can use 6.3. The converse assertion follows from 6.5.

6.7. <u>Corollary</u>. Let Q be a quasimodule with $o(Q) \leq 3$ and let P be a subquasimodule of Q. Then $o(P) \leq 4$. Moreover, if P is not associative then $o(P) = 3$.

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