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A NOTE ON SEPARATION OF SETS BY APPROXIMATELY  
CONTINUOUS FUNCTIONS  
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Abstract: An example of two  $G_\delta$ -sets with disjoint closures in density topology, which cannot be separated by any approximately continuous function is given.

Key words: Density topology, d-derivative of a set, separation properties of approximately continuous functions.

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According to Z. Zahorski [4] given any  $G_\delta$  d-closed (i.e. closed in the density topology) set  $A \subset \mathbb{R}$  there exists a bounded approximately continuous function  $f$  such that  $A = \{x: f(x) = 0\}$ . Consequently, for every pair  $A, B$  of disjoint  $G_\delta$  d-closed sets there is an approximately continuous function  $f$ , which separates  $A$  and  $B$  in the sense that  $0 \leq f \leq 1$ ,  $f = 0$  on  $A$ ,  $f = 1$  on  $B$ .

The last assertion is not generally true, if we suppose  $A, B$  to be  $G_\delta$  sets with disjoint d-closures only, as follows from the example, given in this paper. This answers negatively to the problem posed by M. Laczkovich [2].

Denote by  $\lambda$  the Lebesgue measure and by  $\lambda^*$  the corresponding outer measure. If  $E \subset \mathbb{R}$  is an arbitrary set

and  $x \in \mathbb{R}$ , then we define the outer density of  $A$  at  $x$  by

$$D(E, x) = \overline{\lim} \frac{\lambda^*(\langle x-h, x+h \rangle \cap E)}{2h}$$

and the inner density by  $d(E, x) = 1 - D(\mathbb{R} - E, x)$ . The collection of all sets having the inner density one in each its point forms topology, which will be called the density topology (d-topology). It is easy to see that the d-derivative of a set  $E \subset \mathbb{R}$  will be the set  $\mathcal{D}E = \{x \in \mathbb{R} : D(E, x) > 0\}$  and the d-closure of  $E$  will be  $E \cup \mathcal{D}E$ .

Lemma 1. For an arbitrary bounded interval  $I = (a, b)$  and  $c \in (0, 1)$  there is an open set  $G(I, c) \subset I$  with the following properties:

- (1)  $\{a, b\} \subset \mathcal{D}G(I, c)$ .
- (2) If  $x \in \mathbb{R} - I$  and  $h > 0$ , then  $\lambda(G(I, c) \cap \langle x-h, x+h \rangle) \leq 2ch$ .

Proof. Put

$$d_n = c((n+1)^{-1} - (n+2)^{-1}),$$

$$L = \left[ \bigcup_{n=1}^{\infty} (a+n^{-1} - d_n, a + n^{-1}) \right] \cap (a, a + \frac{1}{2}c(b-a)),$$

$$R = \left[ \bigcup_{n=1}^{\infty} (b-n^{-1}, b - n^{-1} + d_n) \right] \cap (b - \frac{1}{2}c(b-a), b),$$

$$G(I, c) = L \cup R.$$

The property (1) is evidently satisfied, concretely

$$D(G(I, c), a) = D(G(I, c), b) = \frac{1}{2}c$$

(choose  $h = n^{-1}$ ,  $n = 1, 2, \dots$ ). We shall prove (2) for  $x \notin a$ .

We claim

- (3)  $\lambda(\langle x-h, x+h \rangle \cap L) \leq ch$ .

Indeed, consider  $m \in \mathbb{N}$ ,  $(m+1)^{-1} \leq h \leq m^{-1}$ . Then

$$\lambda(\langle x-h, x+h \rangle \cap L) \leq \lambda(\langle a-h, a+h \rangle \cap L) \leq \lambda(\langle a-m^{-1}, a+m^{-1} \rangle \cap L) = c(m+1)^{-1} \leq ch.$$

On the other hand,

$$(4) \quad \lambda(\langle x-h, x+h \rangle \cap R) \leq ch.$$

If  $h < \frac{1}{2}(b-a)$ , then (4) holds trivially, since

$$\langle x-h, x+h \rangle \cap R = \emptyset.$$

If  $h \geq \frac{1}{2}(b-a)$ , then

$$\lambda(\langle x-h, x+h \rangle \cap R) \leq \lambda R \leq \frac{1}{2} c(b-a) \leq ch.$$

From (3) and (4) we immediately obtain (2).

Denote by  $C$  the Cantor's discontinuum (or an arbitrary perfect nowhere dense set with  $\lambda C = 0$ ,  $\inf C = 0$ ,  $\sup C = 1$ ). There are open disjoint intervals  $(a_i, b_i)$  ( $i = 1, 2, \dots$ ) such that

$$C = \langle 0, 1 \rangle - \bigcup_{i=1}^{\infty} (a_i, b_i).$$

The set  $\bigcup_{i=1}^{\infty} (a_i, b_i)$  will be denoted by  $S$ . Further put  $B = C - S$ . Finally, consider

$$A = \bigcup_{i=1}^{\infty} G((a_i, b_i), 2^{-i}).$$

The sets  $A$ ,  $B$  and  $S$  have the following important properties:

Lemma 2. (i)  $\emptyset A \cap B = \emptyset$ .

(ii)  $S \subset \emptyset A$ .

(iii)  $S$  is not a  $G_\sigma$ .

(iv)  $A$  and  $B$  are  $G_\sigma$  sets with disjoint  $d$ -closures.

Proof. (i) Let  $x \in B$ . Choose  $\varepsilon > 0$ . Find a positive integer  $k$  with  $2^{-k} < \varepsilon$ . There is  $\delta > 0$  such that

$$\langle x - \sigma, x + \sigma \rangle \cap \bigcup_{i=1}^k (a_i, b_i) = \emptyset.$$

For every  $i > k+1$  and  $h$ ,  $0 < h < \sigma$  we have

$$\lambda(\langle x-h, x+h \rangle \cap G((a_i, b_i), 2^{-i}) \leq 2^{-i+1}h.$$

Thus

$$\lambda(\langle x-h, x+h \rangle \cap A) \leq \sum_{i=k+1}^{\infty} 2^{-i+1}h < 2\varepsilon h.$$

Since  $\varepsilon > 0$  may be chosen arbitrary,  $x \notin \mathcal{D}A$ .

(ii) For every  $i = 1, 2, \dots$  we obtain from (1)

$$\{a_i, b_i\} \subset \mathcal{D}G((a_i, b_i), 2^{-i}) \subset \mathcal{D}A.$$

(iii) The set  $S$  is of the first category and dense in the Baire space  $C$ , and thus it is not a  $G_\sigma$ .

(iv) Obviously,  $A$  is open and  $\langle 0, 1 \rangle - B = \bigcup_{i=1}^{\infty} \langle a_i, b_i \rangle$ .

Since  $\lambda B = 0$ , we have  $\mathcal{D}B = \emptyset$ . Clearly,  $A \cap B = \emptyset$  and using (i) we obtain

$$c\ell_{\mathcal{D}}A \cap c\ell_{\mathcal{D}}B = c\ell_{\mathcal{D}}A \cap B = \mathcal{D}A \cap B = \emptyset.$$

We shall show that there exists a set whose  $d$ -derivative is not a  $G_\sigma$ :

Theorem 1. If  $A$  is as above, then  $\mathcal{D}A$  is not a  $G_\sigma$ .

Proof. It is an easy consequence of Lemma 2, parts (i), (ii) and (iii).

Definition. A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is said to be approximately continuous if for every  $x \in \mathbb{R}$  there is a set  $M$  such that  $x \in M$ ,  $d(x, M) = 1$  and  $f|_M$  is continuous at  $x$ .

The approximately continuous functions are just the continuous mappings from the density topology to the euclidean

one. Thus,

$$(5) \quad \{x:f(x) = 0\} \text{ is } d\text{-closed}$$

for any approximately continuous function  $f$ .

Since any approximately continuous function  $f$  is of the Baire class one (see for example [3]), it follows that

$$(6) \quad \{x:f(x) = 0\} \text{ is a } G_\delta.$$

Theorem 2. Assume that an approximately continuous function  $f$  vanishes on  $A$ . Then there exists  $x \in B$  with  $f(x) = 0$ . ( $A, B$  are as above.)

Proof. Denote  $M = \{x:f(x) = 0\}$ . By (5),  $\mathcal{D}A \subset M$  and by (6),  $M$  is a  $G_\delta$ . Thus  $\mathcal{D}A \cap C \neq M \cap C$  according to Theorem 1. Hence there is a point  $x \in M \cap C - \mathcal{D}A \subset C - S = B$ .

Corollary. The sets  $A$  and  $B$  cannot be separated by any approximately continuous function.

Remark. It is not difficult to prove that the  $d$ -derivative of any set is always a  $G_\delta$ . We have seen that the  $d$ -derivative need not be a  $G_\delta$ . On the other hand, it need not be a  $F_\sigma$  as well. Indeed, let  $M$  be a measurable set such that  $\lambda(I \cap M) > 0$  and  $\lambda(I - M) > 0$  for every interval  $I$ . Then either  $M$  or  $\mathbb{R} - M$  is not a  $F_\sigma$ . Let us remark only, that if  $M$  is a set whose  $d$ -derivative is not a  $F_\sigma$ , then the upper symmetric derivative of the function  $x \mapsto \lambda(\langle 0, x \rangle \cap M)$  is not of the first class of Baire (although the upper derivative, or even the upper symmetric derivative of arbitrary function is of the second class of Baire, see e.g. [1]).

R e f e r e n c e s

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