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SYMMETRIC EMBEDDING OF FINITE LATTICES INTO FINITE PARTITION LATTICES

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Abstract: It has been shown that every finite lattice can be embedded into a finite partition lattice. Here we show some additional properties which such an embedding can have.

Key words: Finite lattice, partition lattice, symmetric graph, matching.

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For a finite lattice L define the dimension function on L, $d: L \longrightarrow \mathbb{N}$, $d(x)$ = the length of the longest maximal chain between 0 and x. Let Δ denote the kernel of d, let $x \sim y$ denote that there is $\alpha \in \operatorname{Aut}(L)$ such that $x = \alpha(y)$. It is known that in a partition lattice $\Pi(X)$ two partitions are in the relation $\sim$ iff they are of the same type iff they are isomorphic. The partition $\sim$ of L is a refinement of Δ.

Let $\varphi: L \longrightarrow \Pi(X)$ and let θ be the co-image of $\Delta_{\Pi(X)}$, (or $\sim_{\Pi(X)}$), i.e.

$x \,\theta\, y$ iff $d(x) = d(y)$,

(or $x \,\theta\, y$ iff $\exists \alpha \in \operatorname{Aut}(L)$ $x = \alpha(y)$).

Then, clearly, θ satisfies the following two properties

(1) $x \theta y$, $x \leq y \Longrightarrow x = y$, i.e. every class of θ is a co-chain,

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x) This is a part of the CSc dissertation of the author.

(2) for no $x,y,z,t \in L$, $x\,\theta\,y$, $z\,\theta\,t$, $x < z$, $y > t$.

Theorem. If a finite lattice L and a partition θ of L satisfy (1),(2), then there is an embedding of L into some finite $\Pi(X)$ such that

$$x\,\theta\,y \Rightarrow \varphi(x) \sim \varphi(y),$$

$$\neg\, x\,\theta\,y \Rightarrow \neg\,\varphi(x)\,\Delta\,\varphi(y).$$

Corrollary.

1) For every finite lattice, there is an embedding into a finite partition lattice which preserves Δ.

2) The same for $\sim$.

Problem. Let L be a finite lattice and $d': L \longrightarrow \mathbb{N}$ an arbitrary mapping such that $d'(x) < d'(y)$ whenever $x<y$. Is there always an embedding $\varphi: L \longrightarrow \Pi(X)$, X finite, such that, for $y \neq 0$,

$$\frac{d'(x)}{d'(y)} = \frac{d(\varphi(x))}{d(\varphi(y))},$$

where d is the dimension on $\Pi(X)$?

Proofs

Lemma 1: Let $(L_\iota)_{\iota \in I}$ be a system of lattices with the following properties:

1) $|L_\iota \cap L_\kappa| \leq 1$, for $\iota \neq \kappa$,

2) if $x \in L_\iota \cap L_\kappa$ and $y \in L_\iota \cap L_\lambda$ then $x = y$ or x and y are incomparable,

3) if G is the symmetric graph on I, in which (ι, κ) is an edge iff $|L_\iota \cap L_\kappa| = 1$, then G does not contain cycles of length < 5.

Then adding the biggest and the smallest element to $\bigcup_I L_\iota$ we pbtain a lattice.

Proof: The proof of this lemma is just a tedious verification of basic properties of a lattice, we leave it to the reader. (Condition 3) enables us to treat the case such that for some $x \in L_\imath$, $y \in L_\kappa$, where distance of $\imath, \kappa$ in G is 2, there is a nontrivial upper (or lower) bound z. Then we can derive that z must be in L_λ, where λ is uniquely determined by the fact that $(\imath, \lambda)$ and (λ, κ) are edges of G.)

Lemma 2: For every $k \geq 1$, there is a symmetric graph G such that

1) G is bipartite,
2) G can be decomposed into k disjoint matchings,
3) G does not contain cycles of length < 10.

Proof: In [3] a graph $G_{n,m}$ is constructed for all $m,n \geq 2$, which can be decomposed into n disjoint Hamiltonian cycles, does not contain cycles of length $< m$, and is bipartite. Since $G_{n,m}$ is bipartite, the Hamiltonian cycles can be decomposed into matchings, then we can omit superfluous matchings. (Use of the result [3] was suggested by V. Rödl .)

Let $C, D \subseteq L$ be two co-chains in a lattice L. We shall say that they are non-crossing iff for no $x, y \in C$ and $z, t \in D$, $x < z$, $y > t$. A partition θ of L satisfies (1).(2) iff the classes of θ are pairwise non-crossing co-chains.

Lemma 3: Let $C_1, \ldots, C_n$ be a system of non-crossing co-chains of a finite lattice L. Then there is a finite lattice K, and a system of embeddings $\varphi_\imath : L \longrightarrow K$, $\imath \in I$, and for every i, $1 \leq i \leq n$, $x, y \in C_i$, there is a permutation π of the set of indexes I such that

$$\varphi_\imath(x) = \varphi_{\pi(\imath)}(y) \text{ for every } \imath \in I.$$

Proof:

1) $n = 1$. Let $k = |C_1|$ and let $G = (Z,R)$ be the graph of Lemma 2 for k. Let $Z = Z_1 \cup Z_2$ and $R = \bigcup_{x \in C_1} R_x$ be the decompositions given by 1),2) of Lemma 2. Take a system of distinct copies of L, say, $L_\imath$, $\imath \in Z_1$, such that they are also distinct from Z_2. Then glue together $x_\imath$ of $L_\imath$ with κ, for every $x \in C_1$ and $(\imath, \kappa) \in R_x$. Since G does not contain cycles of length < 10, we can use Lemma 1 to obtain a lattice K. For $x,y \in C_1$, the permutation π can be defined putting $\pi(\imath)$ equal to the unique $\kappa \in Z_1$ such that there is $\lambda \in Z_2$, $(\imath, \lambda) \in R_x$, and $(\lambda, \kappa) \in R_y$.

2) $n > 1$. By induction over n, using 1). We have only to add to the induction hypothesis the condition that any co-chain non-crossing with $C_1,\dots,C_n$ is mapped by $\varphi_\imath$, $\imath \in I$, on a co-chain in K.

Proof of the Theorem: Let L, θ satisfy conditions (1), (2), L finite. Let $C_1,\dots,C_n$ be all the classes of the partition θ. Extend L to L' and C_i to C_i', $i = 1,\dots,n$, in such a way that for every two different C_i', C_j' there are $x_0 \in C_i'$, $y_0 \in C_j'$, x_0 comparable with y_0. Let K be the lattice given by Lemma 3 for L', $C_1',\dots,C_n'$, let $\psi: K \longrightarrow \Pi(X)$ be an embedding of K into a finite partition lattice. Take a systen of sets $X_\imath$, $\imath \in I$ of the same cardinality as X, and let $\psi_\imath: K \longrightarrow \Pi(X_\imath)$, $\imath \in I$, be some isomorphic copies of $\psi: K \longrightarrow \Pi(X)$, Finally, define $\varphi: L \longrightarrow \Pi(Y)$, $Y = \bigcup_I X_\imath$, by

$$\varphi(x) = \bigcup_I \psi_\imath(\varphi_\imath(x)).$$

Clearly, φ is an embedding. Now, let $x,y \in C_i'$, then $\varphi_\imath(x) = \varphi_{\pi(\imath)}(y)$ for some permutation π and every $\imath \in I$. Since $\psi_\imath$ and $\psi_{\pi(\imath)}$ are isomorphic, we have

$$\psi_z \varphi_z(x) \sim \psi_{\pi(z)} \varphi_z(x) = \psi_{\pi(z)} \varphi_{\pi(z)}(y).$$

Thus there is a 1-1 correspondence between isomorphic parts of $\varphi(x)$ and $\varphi(y)$, which proves $\varphi(x) \sim \varphi(y)$.

On the other hand, if x,y belong to different classes C_i', C_j', we have $x_0 \in C_i'$, $y_0 \in C_j'$, x_0, y_0 comparable. Then, of course, $\varphi(x_0)$ and $\varphi(y_0)$ must have different dimension. Therefore $\varphi(x)$ and $\varphi(y)$ have different dimension.

The only thing that remains to do now is to take the restriction of φ to L.

R e f e r e n c e s

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