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FUNCTIONALS WITH LINEAR GROWTH IN THE CALCULUS OF VARIATIONS – II

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This part is the direct continuation of the preceding paper in this issue.

3. About the regularity theory. It is well known that generalized BV solutions to the non-parametric Plateau problem are locally Lipschitz continuous and consequently analytic functions. As the two following examples show, the minimum points of our functionals may be non-smooth; in fact they may have jumps not only on the boundary but also on the interior of their domain, and therefore they cannot be $H^{1,1}$ functions.

So if we want to obtain some regularity result, we have to restrict the class of functionals to be considered. In fact we will prove that minimum points are Lipschitz continuous and therefore smooth for functionals of the kind of the area (we are giving below the exact conditions).

Conditions are the ones of O.A. Ladyzhenskaya and N.N. Ural'tseva, see [13], and roughly speaking we can say that they are the ones that grant the a priori estimate of the gradient. We do not show that these conditions are necessary, but, as example 3.2 shows, if these conditions are not satisfied, then solutions may have jumps on the interior.

Our examples concern the 1-dimensional problem, but it is not difficult to extend them to any dimension, for example in a ring.

Example 3.1. Let $u \in BV(-1,1)$ be a generalized solution to problem

$$\begin{cases} \int_{-1}^{1} \sqrt{1 + \alpha(t)\, \dot{u}^2}\, dt \longrightarrow \min \\ u(-1) = -a, \quad u(1) = a \end{cases}$$

that is u minimizes in BV(-1,1) the functional

$$\mathfrak{F}[u] = \int_{(-1,1)} \sqrt{1 + \alpha(t)\, \dot{u}^2} + \sqrt{\alpha(-1)}\,|u(-1) + a| + \sqrt{\alpha(1)}\,|u(1)-a|$$

where

$$\alpha(t) = 1 + t^2 \left(\log \frac{2}{|t|}\right)^4$$

and a is a real constant such that

$$a > \int_{-1}^{1} (\alpha(t) - 1)^{-\frac{1}{2}}\, dt\,.$$

First we have

$$u(-1) = -a\,, \qquad u(1) = a\,; \tag{3.1}$$

to see this, consider the BV(-1,1) function

$$v(t) = \begin{cases} u(t) - u(-1) - a & -1 \leq t < 0 \\ u(t) - u(1) + a & 0 < t \leq 1\,. \end{cases}$$

If (3.1) does not hold, then

$$\mathfrak{F}[v] = \int_{(-1,1)\setminus\{0\}} \sqrt{1 + \alpha(t)\dot{v}^2} + \sqrt{\alpha(0)}\,|u_-(0) - u_+(0) - u(-1) + u(1) - 2a| < \int_{(-1,1)\setminus\{0\}} \sqrt{1 + \alpha(t)\dot{v}^2} + \sqrt{\alpha(0)}\,|u_-(0) - u_+(0)| + \sqrt{\alpha(-1)}\,|u(-1) + a| + \sqrt{\alpha(1)}\,|u(1) - a| = \mathfrak{F}[u]$$

that is we reach a contradiction.

On the other hand u cannot belong to $H^{1,1}(-1,1)$. In this case, in fact, we could define from the Euler equation that

$$u(t) = \frac{\lambda}{\alpha(t)\sqrt{\alpha(t) - \lambda^2}} \qquad \text{a.e. in } (-1,1)$$

for some $\lambda \in \mathbb{R}$ with $\lambda^2 \leq \min\limits_{[-1,1]} \alpha$, hence the contradiction

$$a = \frac{1}{2} u(-1) - u(1) / \leq \int_{-1} |\mathring{u}| dt \leq \int_{-1} \frac{dt}{\sqrt{\alpha(t) - 1}} < a .$$

At this point we have proved that the minimum point u takes the boundary datum and does not belong to $H^{1,1}(-1,1)$. To complete our example we want now to prove that u has a jump exactly in zero, that is; that the singular part of the measure $\mathring{u}$ in the Lebesgue decomposition has support in zero.

Let $(\mathring{u}_R,\mathring{u}_S)$ be the Lebesgue decomposition of $\mathring{u}$ with respect to the Lebesgue measure $\mathcal{L}^1$. Consider the BV(-1,1) function v characterized by

$$\mathring{v}_R = \mathring{u}_R \qquad v(-1) = u(-1)$$

$$\mathring{v}_S = \int_{-1}^{1} \mathring{u}_S \cdot \delta_0$$

where δ_0 is the Dirac measure with support in 0. Then v(1) = = u(1) and the following estimate holds:

$$\int_{-1}^{1} \sqrt{1 + \alpha \mathring{v}^2} = \int_{-1}^{1} \sqrt{1 + \alpha \mathring{v}_R^2} + \sqrt{\alpha(0)} \mid \int_{-1}^{1} \mathring{u}_S \mid \leq$$

$$\leq \int_{-1}^{1} \sqrt{1 + \alpha \mathring{v}_R^2} + \sqrt{\alpha(0)} \int_{-1}^{1} | \mathring{u}_S |$$

Now, since u is a minimum point, we deduce

$$\int_{-1}^{1} \sqrt{\alpha} \mid \mathring{u}_S \mid \leq \sqrt{\alpha(0)} \int_{-1}^{1} | \mathring{u}_S |$$

i.e. supp $\mathring{u}_S \subseteq \{0\}$. On the other hand from the above conside-

ration it is clear that $\mathring{u}_S$ is not the null measure.
Finally note that the above example does not work if we suppose $\alpha(t) \in C^{1,1}(-1,1)$, still with a minimum in zero (in fact we have in this case

$$\int_{-1}^{1} (\alpha(t) - \alpha(0))^{-\frac{1}{2}}\, dt = +\infty \quad),$$

while jumps on the boundary may still occur with smooth $\alpha(t)$ with minimum on the boundary.

Example 3.2. Let $u \in BV(-1,1)$ be a generalized solution to problem

$$(3.2) \qquad \begin{aligned} &\int_{-1}^{1} (1 + \alpha(t)|\mathring{u}|^k)^{1/k}\, dt \longrightarrow \min \\ &u(-1) = -a \qquad u(1) = a \end{aligned}$$

where $k > 2$, $\alpha(t) = 1 + t^2$ and a is a constant greater than

$$\int_{-1}^{1} [(1 + t^2)^{k/k-1} - 1]^{-1/k}\, dt < +\infty .$$

Exactly as in example 3.1 it is easily seen that the solution is smooth in $(-1,1) \setminus \{0\}$, takes boundary datum and has a jump in zero.
Note that this time the obstruction to regularity does not depend on regularity of $\alpha(t)$.
The Bernstein genre of the Euler equation of functional in (3.2) is k. Therefore this example shows that the Dirichlet problem for equations with genre greater than two is generally not solvable on arbitrary domains (see [19]).

We now state the exact hypothesis (see [13]) under which we will prove regularity. We will suppose that Ω be bounded Lipschitz domain, $f(x,p)$ be a function of class $C^2(\bar{\Omega} \times \mathbb{R}^m)$ and $g(x,u)$ be a function of class $C^2(\bar{\Omega} \times \mathbb{R})$. Also we assume that there exist the f_{pxx}, f_{ppx} derivatives and that the following holds:

$$(3.3) \qquad \nu |p| \leq f(x,p) \leq M(1 + |p|)$$

$$(3.4) \qquad |g_u| + |f_p| \leq c$$

$$f_{p_i}(x,p)\, p_i \geq \nu_1 \sqrt{1 + |p|^2} - \nu_2$$

$$(3.5) \qquad |f_{px}| + |f_{ppx}| + |g_{ux}| \leq c$$

$$|f_{p_i px}(x,p)\, p_i| \leq \frac{c}{p}$$

$$(3.6) \qquad g_{uu} \geq 0$$

$$\mu_1 \left(|\xi|^2 - \frac{(\xi, p)^2}{1 + |p|^2} \right) \leq \sum_{i,j=1}^{n} f_{p_i p_j}(x,p)\, \xi_i \xi_j \leq$$

$$\leq \mu_2 \left(|\xi|^2 - \frac{(\xi, p)^2}{1 + |p|^2} \right)$$

where ν, M, ν_1, μ_1 are positive constants, c and ν_2 are non-negative constants, and (ξ, p) denotes the scalar product in $\mathbb{R}^n$.

Finally we suppose that $g(x,u)$ is such that there exist generalized solutions to problem (1.9) (see the end of paragraph 2).

Conditions (3.3)...(3.6) are verified for example by the area or mean curvature functionals, by the functional in example 3.1 if we suppose $\alpha(t) \in C^2(-1,1)$, but they are not satisfied by the functional in example 3.2.

Remark that functionals which satisfy (3.3)...(3.6) have Bernstein's genre equal two, while the general functionals in paragraph 2, i.e. functionals for which only (3.3) holds, may have for example as Bernstein's genre any α with $\alpha > 1$.

Finally observe that (3.6) implies that problem

$$\begin{cases} F[u] = \int_\Omega \{ f(x,Du) + g(x,u)\} \, dx \longrightarrow \min \\ u - \varphi \in H_0^{1,1}(\Omega) \end{cases}$$

has at most one solution in $H^{1,1}(\Omega)$.

We recall the notation

$$\mathcal{F}[u] = \int_\Omega \bar{f}(x, \frac{dx}{d\mu}, \frac{dDu}{d\mu}) \, d\mu + \int_{\partial\Omega} \bar{f}(x, 0, \nu(\varphi - u_{|\partial\Omega})) \, d\mathcal{H}^{n-1} + \int_\Omega g(x,u) \, dx$$

for every $u \in BV(\Omega)$ and $\varphi \in L^1(\partial\Omega)$; and every time we will want to emphasize the domain Ω where the functionals are considered we shall put the subscript Ω to $\mathcal{F}$ or F. Note that for every $u \in H^{1,1}(\Omega)$ with $u = \varphi$ on $\partial\Omega$ we have $\mathcal{F}[u] = F[u]$.

Theorem 3.3. *Let φ belong to $L^1(\partial\Omega)$. Under the above hypotheses, every generalized solution $u \in BV(\Omega)$ to problem*

$$\text{(3.7)} \qquad \begin{cases} \mathcal{F}[u] \longrightarrow \min \\ u \in BV(\Omega) \end{cases}$$

is a locally Lipschitz continuous function.

The idea of the proof is taken from Gerhardt [7]. We first prove the following

Theorem 3.4. *Under hypotheses of theorem 3.3 there exists at least one solution to problem (3.7) which is Lipschitz continuous.*

Theorem 3.5. *Let $\bar{B}_R(x_0)$ be an open ball with radius R small enough. Let φ be a smooth function on ∂B_R. Then there exists at least one solution to problem*

$$\mathcal{F} B_R[u] \longrightarrow \min$$
$$u \in BV(\Omega)$$

<u>which is Lipschitz continuous</u> on $\overline{B}_R$ <u>and such that the trace of</u> u <u>is equal to</u> φ.

Then, as in [7], we will prove theorem 3.3; and, since we have just to change the functional mean curvature with $\mathcal{F}$ in [7] to obtain the proof, we will omit it.

We get theorems 3.4 and 3.5 by an approximation argument using precise estimates (obtained with the same technique of [13]) of the gradients on solutions of approximant problems, the barrier technique and a devise in [9].

<u>Proof of theorem 3.4</u>. Let $u \in BV(\Omega)$ be a solution to problem (3.7) and let $\Omega^* \supset\supset \Omega$. Extend u to $\tilde{u} \in BV(\Omega^*)$ in such a way that

$\int_{\partial\Omega} |D\tilde{u}| = 0$ and consider (see remark 1.7) a sequence $\{u_h\} \subset C^\infty(\mathbb{R}^n)$ such that

$$F_\Omega[u_h] \longrightarrow \int_\Omega \overline{f}(x, \frac{dx}{d\mu}, \frac{dDu}{d\mu})\, d\mu + \int_\Omega g(x,u)\, dx$$
$$u_{h|\partial\Omega} \longrightarrow u_{|\partial\Omega} \qquad \text{in } L^1(\partial\Omega)$$
$$u_h \longrightarrow u \qquad \text{in } L^1(\Omega^*).$$

For every $h \in \mathbb{N}$ and every ε with $0 < \varepsilon < 1$, the approximant problem

$$F_{\Omega^*}[v] + \varepsilon \int_{\Omega^*} |Dv|^2 \longrightarrow \min$$
$$v \in H^{1,2}(\Omega^*)$$
$$v = u_h \quad \text{on } \Omega^* \setminus \Omega$$

has a unique solution $z_{h,\varepsilon} \in H^{1,2}(\Omega^*) \cap C^2(\Omega)$ and we have

(3.8) $\quad \varepsilon\int_{\Omega^*}|Dz_{h,\varepsilon}|^2 + F_{\Omega^*}[z_{h,\varepsilon}] \le \varepsilon\int_{\Omega^*}|Du_h|^2 + F_{\Omega^*}[u_h]$

so that

$$\varepsilon\int_{\Omega^*}|Dz_{h,\varepsilon}|^2, \quad \varepsilon\int_{\Omega^*}|z_{h,\varepsilon}|^2, \quad \int_{\Omega^*}|Dz_{h,\varepsilon}| \quad , \quad \int_{\Omega^*}|z_{h,\varepsilon}|$$

are estimated by a constant which depends on u_h, but does not depend on ε . On the other hand, if $K \subset\subset \tilde{K} \subset\subset \Omega$ we have

(3.9') $\quad \sup_K |z_{h,\varepsilon}| \le c(\operatorname{dist}(K,\partial\tilde{K}), \varepsilon\int_K |z_{h,\varepsilon}|^2, \int_{\tilde{K}} |z_{h,\varepsilon}|)$

(the proof of this kind of estimates is standard, so we will omit it), and

(3.9) $\quad \sup_K |Dz_{h,\varepsilon}| \le c(\operatorname{dist}(K,\partial\tilde{K}), \sup_{\tilde{K}} |z_{h,\varepsilon}|)$

(we delay the proof of (3.9) until later).

Therefore, passing eventually to a subsequence, for ε going to zero we see that $\{z_{h,\varepsilon}\}$ converges weakly in $BV(\Omega^*)$ to some $z_h \in BV(\Omega^*)$ which is locally Lipschitz continuous in Ω and equal to u_h on $\Omega^* \setminus \Omega$. Also

$$\varepsilon\int_{\Omega^*}|Dz_{h,\varepsilon}|^2 \longrightarrow A_h \in \mathbb{R} \quad , \quad A_h \ge 0$$

$$F_{\Omega^*}[z_h] \le \liminf_{\varepsilon\to 0} F_{\Omega^*}[z_{h,\varepsilon}]$$

and passing to the limit in (3.8) for ε going to zero we get

$$A_h + F_{\Omega^*}[z_h] \le F_{\Omega^*}[u_h] .$$

From the last inequality we now deduce that

$\| z_h \|_{BV(\Omega^*)}, A_h \le$ constant which does not depend on h

and consequently

$\sup_K |z_h|, \sup_K |Dz_h| \le$ constant which does not depend on h.

Hence, passing eventually to a subsequence, we may go to the limit for h going to infinity and infer that $\{z_h\}$ converges weakly to some $z \in BV(\Omega^*)$ which is locally Lipschitz continuous in Ω and equal to $\tilde{u}$ in $\Omega^* \setminus \Omega$. Moreover we have

$$A_h \longrightarrow A \in \mathbb{R}$$

$$\int_{\Omega^*} \bar{f}(x, \frac{dx}{d\mu}, \frac{dDz}{d\mu})\, d\mu + \int_{\Omega^*} g(x,z) + A \leq$$

$$\int_{\Omega^*} \bar{f}(x, \frac{dx}{d\mu}, \frac{dD\tilde{u}}{d\mu})\, d\mu + \int_{\Omega^*} g(x,\tilde{u})$$

from which, taking into account convexity and homogeneity of $\bar{f}$, we deduce that $A = 0$ and z is a generalized solution to problem (3,7). q.e.d.

We now pass to prove 3.9. In fact, we shall prove a sharper a priori estimate of the gradients and precisely

<u>Proposition 3.6.</u> <u>Let</u> $\lambda \leq 1$ <u>be a non-negative real constant and</u> $\psi \in H^{1,2}(\Omega)$, <u>let</u> u_λ <u>be a solution to problem</u>

$$(3.10) \qquad \begin{cases} F_\Omega[v] + \lambda \int_\Omega |Dv|^2 \longrightarrow \min \\ v \in H^{1,2}(\Omega) \qquad v - \psi \in H_0^{1,2}(\Omega)\,. \end{cases}$$

<u>Then for every ball</u> B_R <u>with</u> $B_{2R} \subset\subset \Omega$

$$\| Du_\lambda \|_{\infty, B_{R/4}} \leq C$$

<u>where</u> C <u>depends on</u> $\frac{1}{R} \operatorname{osc}_{B_{2R}} u_\lambda$ <u>and does not depend explicitly on</u> λ .

The fact of emphasizing the dependence on $\frac{1}{R} \operatorname{osc}_{B_{2R}} u_\lambda$ in the gradients estimate is just what will allow us to prove theorem 3.5.

The following lemma due to Stampacchia [21] and the below Sobolev estimate, which can be obtained with the same proof in [13], will turn out useful.

Lemma 3.7. *Let $u(h,\ell)$ and $a(h,\ell)$ be non-negative functions defined for $h > 0$ and $0 < \ell \leq R_0$, increasing in ℓ for h fixed and decreasing in h for ℓ fixed. Suppose*

$$u(h,\ell) \leq \frac{c_1}{(R-\ell)^2}\, u(k,R) \quad a(k,R)^{2/n}$$

$$a(h,\ell) \leq \frac{1}{(h-k)^2}\, u(k,R)$$

where $h > k \geq 0$, $0 < \ell < R \leq R_0$. Then there exists d such that

$$a(d,R_0/2)u(d,R_0/2) = 0$$

and

$$d \leq C_2(c_1,n)\; u(0,R_0)^{1/2} R_0^{-\frac{n}{2}\theta}\; a(0,R_0)^{\frac{\theta-1}{2}}$$

where

$$\theta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{n}}\ .$$

Lemma 3.8. *Set*

$$S_\lambda = \{(x,x_{n+1}) \in \Omega \times \mathbb{R}^n : x_{n+1} = u_\lambda(x)\}.$$

Then for every $g \in C_0^\infty(B_R)$

$$\left(\int_{S_\lambda} g^{\frac{2n}{n-2}}\, d\mathcal{H}^n\right)^{\frac{n-2}{2n}} \leq C_3 \left(\int_{S_\lambda} |\delta g|^2\, d\mathcal{H}^n\right)^{1/2}$$

where C_3 depends on $\operatorname{osc}_{B_R} u_\lambda$.

Proof of proposition 3.6: the proof follows the one in [13]. Putting $\varphi = (u(x) - u(x_0))\,\xi^2$ with $\xi \in C_0^\infty(B_{2R}(x_0))$, $\xi = 1$ on $B_R(x_0)$, $|\nabla \xi| \leq c/R$ as test function in the Euler equation for the functional in (3.10) we deduce

$$(3.11)\quad \frac{1}{R^n}\left\{\int_{B_R(x_0)}\sqrt{1+|Du_\lambda|^2}+\int_{B_R(x_0)}|Du_\lambda|^2\right\}\le$$

$$\le C\left(1+\frac{1}{R}\operatorname*{osc}_{B_{2R}(x_0)} u_\lambda\right)^2$$

where C does not depend on λ .

Set

$$\omega_\lambda = \log(1+|Du_\lambda|^2)$$

$$A_{\lambda,k} = \{x\in\Omega : \omega_\lambda(x)>k\}$$

$$S_{\lambda,k} = \{(x,u_\lambda(x)) : x\in A_{\lambda,k}\}.$$

Putting in the Euler equation as test function

$$\varphi = D_s\,(D_s u \max(\omega_\lambda - k,0)\,\xi^2)$$

integrating by parts and summing on $s = 1,n$, we deduce .

$$(3.12)\quad \int_{S_{\lambda,k}}|\delta\omega_\lambda|^2\xi^2\,d\mathcal{H}^n+\lambda\int_{A_{\lambda,k}}(1+|Du_\lambda|^2)\,|D\omega_\lambda|^2\xi^2\,dx\le C\int_{S_{\lambda,k}}(\omega_\lambda-k)^2\,|\delta\xi|^2+$$

$$+\lambda\int_{A_{\lambda,k}}(1+|Du_\lambda|^2)\,(\omega_\lambda-k)^2\,|D\xi|^2$$

while putting $\varphi = D_s\,(D_s u \max(\omega_\lambda - k,0)^2\,\xi^2)$ we deduce

$$(3.13)\quad \lambda\int_{A_{\lambda,k}}(\omega_\lambda-k)^2\,|D^2u_\lambda|^2\xi^2\le C\left\{\int_{S_{\lambda,k}}(\omega_\lambda-k)^2(\xi^2+|\delta\xi|^2)+\lambda\int_{A_{\lambda,k}}(1+|Du_\lambda|^2)\,|D\xi|^2(\omega_\lambda-k)^2\right\}.$$

Set now

$$A_\lambda(h,\ell) = A_{\lambda,h}\cap B_\ell \qquad S_\lambda(h,\ell) = (A(h,\ell)\times\mathbb{R})\cap S_\lambda$$

$$a_\lambda(h,\ell) = \text{meas}\,A_\lambda(h,\ell)$$

$$u_\lambda(h,\ell) = \int_{S_\lambda(h,\ell)}(\omega_\lambda-k)^2+\int_{A(h,\ell)}(1+|Du_\lambda|^2)(\omega_\lambda-k)^2$$

Using (3.12),(3.13),(3.11) and the Sobolev estimate in lemma 3.8 it is easily verified that hypotheses of lemma 3.7 hold for functions $u_\lambda(h,\ell)$ and $a_\lambda(h,\ell)$ with constant c_1 depending on $\frac{1}{R}\operatorname{osc}_{B_{2R}} u_\lambda$. So that we have

$$\| Du_\lambda \|_{\infty, B_{R/4}(x_0)} \le e^d$$

where d is estimated by a constant depending on $\frac{1}{R}\operatorname{osc}_{B_{2R}} u_\lambda$ multiplied by

$$(3.14) \le \left\{ \frac{1}{R^n} \int_{S_{\lambda,R/2}} \omega_\lambda^2 + \frac{\lambda}{R^n} \int_{B_{R/2}} (1 + |Du|^2)\,\omega_\lambda^2 \right\}^{1/2} .$$

Now to estimate (3.14), put in the Euler equation as test function $[u(x) - u(x_0)]\,\omega_\lambda^2\,\zeta^2$; it is not difficult to deduce

$$(3.14) \quad \left\{ \frac{c}{R^n}\left[(1 + \frac{1}{R}\operatorname{osc}_{B_{2R}} u_\lambda)^2 \int_{B_R} \omega_\lambda^2 + \right.\right.$$
$$\left.\left. + (\operatorname{osc}_{B_{2R}} u_\lambda)^2 \left\{ \int_{S_{\lambda,R}} |\delta\omega_\lambda|^2 \zeta^2 + \lambda \int_{B_R} |D\omega_\lambda|^2 \zeta^2 \right\}\right]\right\}^{1/2}$$

and again choosing as test function $D_s\,(D_s u\cdot\zeta^Q)$ we derive

$$(3.14) \le C\ (\text{depending on } \tfrac{1}{R}\operatorname{osc}_{B_{2R}} u_\lambda) \times$$
$$\times \left[\frac{1}{R^n}\int_{B_{2R}} (1 + |Du_\lambda|^2)^{1/2} + \frac{\lambda}{R^n}\int_{B_{2R}} |Du_\lambda|^2 + \frac{1}{R^n}\int_{B_{2R}} \omega_\lambda^2\right]^{1/2}$$

and finally because of $\omega_\lambda^2 \le \sqrt{1 + |Du_\lambda|^2}$ and (3.11) we get the proof.

Unfortunately we are not able to show that a generalized solution and a smooth solution to problem (3.7) differ for constant. Had we proved this, then theorem 3.3 would easily follow.

Since this is not the case, we have to prove theorem 3.5.

However, this proof also yields us the method to prove global regularity.

Proof of theorem 3.5: remark that theorem 3.5 follows with the same proof as theorem 3.4 if we are able to state global estimates (3.9'), (3.9). Concerning the estimate of the maximum of solution u_λ we have: let u_λ be a solution to problem

$$(3.15)\qquad \begin{aligned} &F_R[u] = \int_{B_R} f(x,Dv) + \int_{B_R} g(x,v) + \lambda \int_{B_R} |Dv|^2 \longrightarrow \min \\ &v - \varphi \in H_0^{1,2}(B_R) \end{aligned}$$

then

$$\| u_\lambda \|_{\infty, B_R} \leq C$$

where C depends on $\| \varphi \|_{\infty, \partial\Omega}$. This is easily seen comparing $F_R[u_\lambda]$ with F_R computed on

$$v(x) = \begin{cases} k & \text{in } A_k = \{x \in \Omega : u_\lambda(x) > k\} \\ u\ (x) & \text{in } x \in \Omega \setminus A_k \end{cases}$$

$k > \max_{\partial\Omega} |\varphi|$, using the hypothesis on g (made to get existence in paragraph 1) which grants estimate essentially of this kind

$$\int_{A_k} |g_u| \; |u_\lambda - k| \leq \nu (1 - \varepsilon_0) \int_{A_k} |Du_\lambda|$$

and the Stampacchia's well known global lemma analogous to lemma 3.7 (see [21]).

Now we pass to consider the gradient estimate. With the barrier technique (see Serrin [19]) it is not difficult to show that for R small enough, there exists a constant K depending on the C^2 norm of φ such that, if u_λ is a solution to (3.15) then

(3.16) $\qquad |u_\lambda(x) - u_\lambda(y)| \leq K|x - y|$

for every $x \in B_R$, every $y \in \partial B_R$ and every λ with $0 \leq \lambda \leq 1$. Now from (3.16) and proposition 3.6 (see theorem 2.1 of [9] or theorem 1.4 of [10]) it follows

$$\| Du_\lambda \|_{\infty, \bar{B}_R} \leq C$$

where C depends on the C^2 norm of φ on ∂B_R and does not depend on λ . q.e.d.

It is now clear that if φ is smooth and one is able to construct barriers relative to the functional F_Ω for an open set Ω , then problem (3.7) has a unique smooth solution on $\bar{\Omega}$ which takes the boundary datum φ . We do not enter this question.

To close, we remark that $x^{1/3} \in H^{1,1}(-1,1)$ is an extremal for the functional

$$F[u] = \int_{-1}^{1} \sqrt{1 + \dot{u}^2} - 3 \int_{-1}^{1} \sqrt{\frac{u^4}{1 + 9u^4}} .$$

This, perhaps, may show the relevance of the hypothesis of convexity on $g(x,u)$.

Note

(1) $F(x, \lambda p + (1-\lambda)q) \leq \lambda F(x,p) + (1-\lambda)F(x,q)$ for every $\lambda \in (0,1)$, the equality sign holding only if p and q lie on the same ray from the origin.

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