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FUNCTIONALS WITH LINEAR GROWTH IN THE CALCULUS OF VARIATIONS – I

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Abstract: In this paper variational problem with generalized area functional is considered. This functional is defined for the BV- functions by the explicit integral formula and the existence of weak solutions is proved. The example of interior nonregularity is given and the sufficient conditions for the interior regularity are found.

Key words: generalized area problem, regularity, weak solutions.

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In this paper we deal with functionals

$$\mathcal{F}[u] = \int_\Omega f(x,Du)dx \tag{0.1}$$

where $f(x,p)$ is a continuous and convex in p function with linear growth, i.e.

$$\nu\,|p| \le f(x,p) \le M(1 + |p|)$$

and we consider the problem of minimizing $\mathcal{F}[u]$ under Dirichlet boundary conditions, i.e.

$$u = \varphi \quad \text{on } \partial\Omega$$

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The area functional, the functional

$$(0.3) \qquad \int_{\Omega} \sqrt{1 + |x|^2} \quad \sqrt{1 + |Du|^2}$$

first considered by D. Bernstein [3], and then by J. Serrin [19], and the functionals

$$\int_{\Omega} (1 + \alpha(x)|Du|^k)^{1/k}\, dx\,, \qquad k > 1$$

are simple examples of functionals we shall consider.

Boundary curvature restrictions are necessary for solvability of the Dirichlet problem for Euler equations of (0,1), in classes of smooth functions, as it was pointed our by Jenkins and Serrin [12] (see also Bernstein [3]), and inquired in general setting by Serrin [19]. Moreover, some structure conditions on $\mathfrak{F}$ have to be imposed (Serrin [19]).

We first deal with the question of defining appropriate generalized solutions when the curvature relations do not hold. In fact we extend the functional (0,1) to the space BV(Ω) of functions whose derivatives are measures with bounded total variation, giving an "integral representation"; then we prove, using a result by Reschetniyak [17], semicontinuity for this extended functional and then existence of generalized BV solutions to the problem (0,1) (0.2). Our extended functional will coincide with the natural semicontinuous functional constructed analogously to the Lebesgue definition of surface area (see Serrin [18]) but, furthermore, our "integral represented" extention will allow us to inquire on the regularity properties of the minimum points.

As a matter of fact we shall study more general functionals of the following kind

$$\int_\Omega [f(x,Du) + g(x,u)]\, dx$$

but we shall not study the general situation

$$(0.4) \qquad \int_\Omega f(x,u,Du)\, dx$$

because we are not able to obtain an "integral represented" extention to BV of the functional (0.4) consistent with the semicontinuity theorem (see the end of paragraph 2). Obviously the Lebesgue analogous extention still holds, but this does not enable us to inquire on regularity of minimum points.

The existence theory for the functional (0.1) has much in common with the one for the non-parametric Plateau problem in $BV(\Omega)$, but as we shall see (examples 3.1 and 3.2), in our more general situation generalized solutions may have jumps not only on the boundary but also on the interior; that is, it may occur that solutions are not $H^{1,1}$ functions.

For a class of functionals, including for example Bernstein's functional (0.3), essentially those for which an a priori estimate of gradients holds, see Ladyzhenskaya and Ural'tseva [13], we are able to show that generalized BV solutions are smooth on the interior of Ω , with jumps still possible on the boundary. Proofs of regularity use an idea by Gerhardt [7] and techniques analogous to the ones in Giaquinta [9], Giaquinta and Modica [10].

Finally we want to remark that many problems are left still open in this setting, for example the study of general case (0.4) and of regularity in more general hypotheses.

1. <u>The space</u> $BV_\Phi(\Omega)$ We begin by recalling some well-known definitions and theorems. The reader may consult [14] or

[2] as general references.

Let Ω be a bounded open subset of $\mathbb{R}^n$, $BV(\Omega)$ is the space of the Lebesgue summable functions, whose derivatives are measure of bounded total variation on Ω. Du and $|Du|$ denote respectively the vector valued measure whose components are the derivatives of u and the measure total variation of Du. $BV(\Omega)$ is a Banach space with the norm

$$\|u\|_{BV(\Omega)} = \int_\Omega |u(x)|\,dx + \int_\Omega |Du|$$

where $\int_\Omega |Du|$ is the total variation on Ω of the measure $|Du|$.

We shall now state a few theorems which will turn useful later on (see [15],[2])

i) The immersion of $BV(\Omega)$ in $L^1(\Omega)$ is compact, i.e. let $\{u_k\}$ be a bounded sequence in $BV(\Omega)$, that is

$$\int_\Omega |u_k| + \int_\Omega |Du_k| \leq M \qquad M \in \mathbb{R}_+$$

then there exists a subsequence which converges in $L^1(\Omega)$ to some function u belonging to $BV(\Omega)$. This is the analogous of the Rellich theorem for Sobolev spaces.

ii) Let Ω be a Lipschitz domain with L as Lipschitz constant. For every $u \in BV(\Omega)$ the trace of u is defined, and we have the following estimate

$$|u|\,d\mathcal{H}^{n-1} \leq \sqrt{1+L^2}\int_\Omega |Du| + c(\Omega)\int_\Omega |u|\,dx$$

Also, the Green formula holds, i.e.

$$\int_\Omega D_i g\, u\, dx + \int_\Omega g\, D_i u = \int_{\partial\Omega} u\, g\, \nu_i\, d\mathcal{H}^{n-1}, \ \forall g \in C_0^\infty(\Omega)$$

where $\nu = (\nu_i)$ is the exterior normal vector to $\partial\Omega$.

iii) More generally, let Σ be a (n-1)-dimensional oriented

Lipschitz manifold in Ω with normal vector ν. For every $u \in BV(\Omega)$ the left trace u^- and the right trace u^+ of u are defined on Σ, and we have

$$D_i u_{|\Sigma} = (u^- - u^+)\,\nu_i \mathcal{H}^{n-1}$$

$$|Du|_{|\Sigma} = |u^- - u^+|\,\mathcal{H}^{n-1}$$

Let us consider a bounded open set Ω^* such that $\Omega^* \supset\supset \Omega$; then every $\varphi \in L^1(\partial\Omega)$ may be extended [6] to $\Omega^* \setminus \Omega$ as a $\phi \in H^{1,1}(\Omega^* \setminus \Omega)$ with $\phi_{|\partial\Omega^*} = 0$.

Definition. For every $\varphi \in L^1(\partial\Omega)$, $BV_\phi(\Omega)$ is the space of functions u in $BV(\Omega^*)$ such that $u = \phi$ in $\Omega^* \setminus \Omega$.

Observe that the space $BV_\phi(\Omega)$ does not depend, in an essential way, on the extentions Ω^* of Ω and ϕ of φ.

From i) and general results in measure theory it follows that from every bounded sequence $\{u_k\}$ in $BV_\phi(\Omega)$ we can extract a subsequence $\{u_{k_i}\}$ such that

$$u_{k_i} \longrightarrow u \quad \text{in} \quad L^1(\Omega^*)$$

$$Du_{k_i} \longrightarrow Du \text{ weakly as vector valued measures.}$$

We shall say that $\{u_{k_i}\}$ converges weakly in $BV(\Omega^*)$ to u. Note that u belongs to $BV_\phi(\Omega)$ and that Du may have support on $\partial\Omega$ even if Du_{k_i} have support on Ω; that is u can have a jump on $\partial\Omega$ even if u_{k_i} have no jumps on $\partial\Omega$.

We could define the space $BV_\phi(\Omega)$ in a different way (see for a general treatment [20]) which, maybe, can make things clearer. For every $\varphi \in L^1(\partial\Omega)$ let us define the space $BV_\phi(\bar{\Omega})$ as the space of functions u for which there exist n measures α_i on $\bar{\Omega}$ such that the Green formula

$$\int_\Omega u\, D_i g + \int_{\bar\Omega} g\alpha_i = \int_{\partial\Omega} u\, g\, \nu_i\,, \qquad \forall g \in C^\infty(\bar\Omega)$$

holds.

Now it is easily seen that $BV_\phi(\bar\Omega)$ is the "restriction" of $BV_\phi(\Omega)$, and $BV_\phi(\Omega)$ is an "extention" of $BV_\varphi(\bar\Omega)$.

2. <u>Existence theorems</u>. Let Ω be a bounded Lipschitz domain and let $f(x,p): \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function which is convex in p and satisfies the condition

(1.1) $\qquad \nu\,|p| \leq f(x,p) \leq M(1+|p|)$

where ν, M are positive constants. Consider the minimum problem

$$(1.2) \qquad \begin{cases} \mathcal{F}[u] = \int_\Omega f(x,Du)\,dx \longrightarrow \min \\ u = \varphi \quad \text{on } \partial\Omega \end{cases}$$

where $\varphi \in L^1(\partial\Omega)$.

The functional $\mathcal{F}[u]$ is well defined in $H^{1,1}(\Omega)$. So we could look for a minimum on the set of $u \in H^{1,1}(\Omega)$, $u = \varphi$ on $\partial\Omega$. But we cannot use direct methods of the calculus of variations, because the space $H^{1,1}(\Omega)$ does not have weakly compact balls; on the other hand, it is well known, think for example of the non-parametric area problem, that there is no hope of obtaining $H^{1,1}$ solutions $u = \varphi$, while, still considering the non-parametric area problem, the appropriate generalized solutions are $BV_\phi(\Omega)$ functions.

We now want to extend the functional $\mathcal{F}[u]$ to a functional $\bar{\mathcal{F}}[u]$ defined on $BV_\phi(\Omega)$ and prove the existence of a minimum point in $BV_\phi(\Omega)$ for $\bar{\mathcal{F}}[u]$.

We define for every $(x,p) \in \bar\Omega \times \mathbb{R}$ and for every $p_0 > 0$

the function

$$\bar{f}(x,p_o,p) = f(x,\frac{p}{p_o})\, p_o$$

Clearly $\bar{f}(x,p_o,p)$ is continuous in $\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}$, convex in (p_o,p), and homogeneous of degree 1 in (p_o,p).

<u>Proposition 1.1</u>. <u>The following is true:</u>

a_1) <u>There exists the limit of</u> $\bar{f}(x,p_o,p)$ <u>for</u> p_o <u>going to</u> 0^+,

a_2) $\bar{f}(x,p_o,p) \leq M(p_o + |p|)$

$\bar{f}(x,p_o,p) \geq \nu\, |p|$

a_3) f <u>is convex in</u> (p_o,p) <u>where</u> $p_o \geq 0$, $p \in \mathbb{R}^n$, <u>and homogeneous of degree</u> 1 <u>in</u> (p_o,p).

<u>Proof</u>: $\bar{f}(x,p_o,p)$ is convex, locally bounded and continuous in (p_o,p); then for every fixed p there exists

$$\lim_{p_o \to 0^+} \bar{f}(x,p_o,p) = \bar{f}(x,0,p)$$

From (1.1) and the convexity of f it follows

$$|f(x,p) - f(x,q)| \leq M\, |p-q|$$

so that

$$|\bar{f}(x,0,p) - \bar{f}(x,0,q)| = \left| \lim_{\lambda \to +\infty} \frac{f(x,\lambda p) - f(x,\lambda q)}{\lambda} \right| \leq M|p-q|$$

and then a_1) is proved. The proofs of a_2) and a_3) are trivial.

q.e.d.

Let us choose a positive Radon measure μ in such a way that $|Du|$ and the Lebesgue measure $\mathcal{L}^n$ be absolutely continuous with respect to μ , and denote by $\frac{dx}{d\mu}$ and $\frac{dDu}{d\mu}$ the Radon-Nikodym derivatives respectively of the Lebesgue measure $\mathcal{L}^n$ and of the vector valued measure Du with respect to the measure μ . We define for every $u \in BV(\Omega)$

(1.3) $\overline{\mathcal{F}}[u] = \int_\Omega \overline{f}(x, \frac{dx}{d\mu}, \frac{dDu}{d\mu})\, d\mu$.

Observe that this is a well posed definition; in fact from the homogeneity of $\overline{f}$ it follows that $\overline{\mathcal{F}}[u]$ does not depend on the choice of the measure μ ; also, clearly, for every $u \in H^{1,1}(\Omega)$

$$\overline{\mathcal{F}}[u] = \mathcal{F}[u] .$$

For example, if $\mathcal{F}$ is the non-parametric area functional, $\overline{\mathcal{F}}$ is the total variation of the vector valued measure $(\mathcal{L}^n, Du)$, which is just the definition of the area for a generalized BV surface.

Now we recall the following semicontinuity theorem due to Reschetniyak [17]

<u>Theorem 1.2.</u> <u>Let</u> $F(x,p): \overline{\Omega} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ <u>be a continuous function which is convex and homogeneous of degree</u> 1 <u>in</u> p. <u>Consider the following functional defined on the space of the vector valued measure with bounded total variation</u>:

$$I[\mu] = \int_\Omega F(x, \frac{d\mu}{d|\mu|})\, d|\mu| .$$

<u>Then for every weakly convergent sequence</u> $\{\mu_k\}$ <u>of measures</u>, i.e. $\mu_k \rightharpoonup \mu_0$ <u>we have</u>

$$\liminf_{k \to \infty} I[\mu_k] \geq I[\mu_0] .$$

At this point we are ready to show that problem (1.2) has a generalized solution in $BV_\phi(\Omega)$. Extend $f(x,p)$ to $\Omega^* \supset\supset \Omega$ as continuous function in (x,p) and convex function in p (we continue to call this extention f); construct the function $\tilde{f}(x,p_0,p)$ and consider the functional, defined on $BV_\phi(\Omega)$, as in (1.3)

$$(1.4) \qquad \overline{\mathcal{F}}_{\Omega^*}[u] = \int_{\Omega^*} \overline{f}(x, \frac{dx}{d\mu}, \frac{dDu}{d\mu})\, d\mu$$

where $\phi \in H^{1,1}(\Omega^* \setminus \Omega)$ with $\phi = \varphi$ on $\partial\Omega$ and $\phi = 0$ on $\partial\Omega^*$. Then we have

Theorem 1.3. There exists a minimum point for the problem

$$(1.5) \qquad \begin{cases} \overline{\mathcal{F}}_{\Omega^*}[u] \longrightarrow \min \\ u \in BV_\phi(\Omega) . \end{cases}$$

Proof: This follows using the standard direct methods of the calculus of variation, if we just remark that the functional $\overline{\mathcal{F}}$, from Theorem 1.2, is lower semicontinuous with respect to the weak convergence in $BV_\phi(\Omega)$ and

$$\overline{\mathcal{F}}_{\Omega^*}[u] \geq \nu \int_{\Omega^*} |Du| \qquad \forall\, u \in BV_\phi(\Omega)$$

and the Poincaré inequality

$$\int_{\Omega^*} |u| \leq c(\Omega) \int_{\Omega^*} |Du|$$

holds. q.e.d.

Now observe that for every u in $BV_\phi(\Omega)$ we have

$$\overline{\mathcal{F}}_{\Omega^*}[u] = \int_\Omega \overline{f}(x, \frac{dx}{d\mu}, \frac{dDu}{d\mu}) d\mu + \int_{\partial\Omega} \overline{f}(x, 0, \frac{dDu}{d\mathcal{H}^{n-1}}) d\mathcal{H}^{n-1} +$$
$$+ \int_{\Omega^* \setminus \Omega} f(x, D\phi)\, dx$$

and that

$$\frac{dDu}{d\mathcal{H}^{n-1}}\Big|_{\partial\Omega} = \nu(\varphi - u_{|\partial\Omega})$$

where ν is the exterior normal vector to $\partial\Omega$.

Then we have proved

Theorem 1.4. For every $\varphi \in L^1(\partial\Omega)$ there exists a minimum point for the functional

$$\int_{\Omega} \bar{f}\left(x,\frac{dx}{d\mu},\frac{dDu}{d\mu}\right)d\mu + \int_{\partial\Omega} \bar{f}(x,0,\nu(\varphi - u_{|\partial\Omega}))\, d\mathcal{H}^{n-1}$$

in $BV(\Omega)$.

These minimum points are the ones which are to be considered the appropriate generalized solutions for problem (1.2). This will clearly show on behalf of the rest of the paper. The two following examples may, however, be a justification.

Let $f(x,p) = \sqrt{1 + |p|^2}$, that is, if we consider the non-parametric area problem, we have $\bar{f}(x,0,\nu(\varphi - u_{|\partial\Omega})) = |u - \varphi|$; if $f(x,p) = \sqrt{a_0(x) + a_{ij}(x)p_i p_j}$ for some definite positive matrix $a_{ij}(x)$ and $a(x) > 0$ we obtain $\bar{f}(x,0,\nu(\varphi - u_{|\partial\Omega})) =$
$= \sqrt{a_{ij}(x)\nu_i \nu_j}\,|\varphi - u|$.

<u>Remark 1.5</u>. Let Σ be an oriented $(n-1)$-dimensional Lipschitz manifold in Ω ; with the notations of iii) of paragraph 1 we have

$$\int_{\Sigma} \bar{f}\left(x,\frac{dx}{d\mu},\frac{dDu}{d\mu}\right)d\mu = \int_{\Sigma} \bar{f}(x,0,\nu(u^+ - u^-))\, d\mathcal{H}^{n-1} .$$

Now we want to relate our functional $\mathcal{F}_{\Omega^*}$ in (1.4) with the one constructed analogously to the Lebesgue definition of surface area (see Serrin [18]), i.e.

$$F[u] = \inf\{\liminf_{k\to\infty} \mathcal{F}[u_k] : u_k \in H_0^{1,1}(\Omega^*) \quad u_k \longrightarrow u \text{ in } L^1(\Omega^*)\} .$$

From the semicontinuity theorem for $\mathcal{F}$ and (1.1) we derive that

$$F[u] \geq \mathcal{F}_{\Omega^*}[u] \qquad \forall\, u \in BV_\phi(\Omega) .$$

On the other hand, it is well known ([1],[2]) that for every $u \in BV_\phi(\Omega)$ there exists a sequence $\{u_k\}$ in $C^\infty(\Omega^*)$ such that

$$u_k \longrightarrow u \quad \text{in} \quad L^1(\Omega^*)$$

$$\int_{\Omega^*} |Du_k| \longrightarrow \int_{\Omega^*} |Du| .$$

Then from the following theorem still due to Reschetniyak [17] it follows that for every $u \in BV_\phi(\Omega)$ there exists a $C^\infty(\Omega^*)$ sequence $\{u_k\}$ such that

$$u_k \longrightarrow u \quad \text{in} \quad L^1(\Omega^*)$$

$$\overline{\mathfrak{F}}_{\Omega^*}[u_k] \longrightarrow \overline{\mathfrak{F}}_{\Omega^*}[u] .$$

<u>Theorem 1.6.</u> <u>Let</u> $\{\mu_k\}$ <u>be a sequence of vector valued measures weakly converging to</u> μ . <u>Suppose that there exists a continuous function</u> $F(x,p)$ <u>which for every</u> x <u>is homogeneous of degree</u> 1 <u>and "strictly"</u>$^{(1)}$ <u>convex in</u> p <u>and such that</u>

$$(1.6) \qquad \int_\Omega F\left(x,\frac{d\mu_k}{d|\mu_k|}\right) d|\mu_k| \longrightarrow \int_\Omega F\left(x,\frac{d\mu}{d|\mu|}\right) d|\mu|$$

<u>Then</u> (1.6) <u>holds for every continuous function</u> $F(x,p)$ <u>homogeneous of degree</u> 1 <u>and convex in</u> p.

<u>Remark 1.7.</u> (See [1].) Suppose that $u \in BV_\phi(\Omega)$ and that $\int_{\partial\Omega} |Du| = 0$. Then for the $C^\infty(\Omega^*)$ sequence $\{u_k\}$ above we have

$$u_k \longrightarrow u \quad \text{in} \quad L^1(\Omega^*)$$

$$\int_\Omega \bar{f}\left(x,\frac{dx}{d\mu},\frac{dDu_k}{d\mu}\right) d\mu \longrightarrow \int_\Omega \bar{f}\left(x,\frac{dx}{d\mu},\frac{dDu}{d\mu}\right) d\mu$$

$$u_{k\,|\partial\Omega} \longrightarrow u_{|\partial\Omega} \quad \text{in} \quad L^1(\partial\Omega)$$

We now want to spend a few words to consider some simple generalizations of the above situation. Let $g(x,u)$ be a function which is measurable in x and continuous in u such that

$$(1.7) \qquad u \longrightarrow \int_\Omega g(x,u)\, dx$$

is a lower semicontinuous functional with respect to the $L^1(\Omega)$

convergence. Moreover suppose that we have an estimate of this kind

(1.8) $$\left| \int_\Omega g(x,u) \right| \leq \nu (1 - \varepsilon_0) \int_\Omega |Du| + c$$

where ε_0 is some positive constant less than one, ν is the constant in (1.1) and c is some absolute constant. Then it is obvious that we obtain a generalized solution for the minimum problem

(1.9) $$\begin{cases} \int_\Omega \{ f(x,Du) + g(x,u) \} \, dx \longrightarrow \min \\ u = \varphi \quad \text{on } \partial\Omega \, . \end{cases}$$

Without going into further detail one of the two following conditions grants the lower semicontinuity of the functional (1.7)

s_1) $|g(x,u)| \leq k(1+|u|)$

s_2) $g(x,u)$ convex in u.

Condition (1.8) is less simply verified. It occurs in an analogous way studying the Dirichlet problem for surfaces of prescribed mean curvature. For example it is satisfied in situation s_1) if we suppose that the constant k is small enough and in situation s_2) when

$$\left\| \lim_{t \to \infty} \max (g_u(x,t),0) \right\|_{L^n(\Omega)} \leq \nu \, n\omega_n^{1/n}$$

$$\left\| \lim_{t \to -\infty} \min (g_u(x,t),0) \right\|_{L^n(\Omega)} \leq \nu \, n\omega_n^{1/n}$$

because of the complete analogy with the Dirichlet problem for surfaces of prescribed mean curvature, we refer to [8],[9],[11] for a discussion of sufficient, and, in some sense, necessary condition to grant the estimate (1.8).

To close this paragraph we want to justify the reason why we do not study the general function

$$\int_\Omega f(x,u,Du)\,dx.$$

If u belongs to $BV(\Omega)$, then (Federer [4],[5]) the limits of the average of u exist $\mathcal{H}^{n-1}$-a.e.; on the other hand $|Du|$ is absolutely continuous with respect to $\mathcal{H}^{n-1}$: so that we can redefine u in such a way that u is a μ-measurable function where, for example, $\mu = \alpha^n + |Du|$. And then we could define the general functional

$$\bar{\mathcal{F}}[u] = \int_\Omega \bar{f}(x,u,\frac{dx}{d\mu},\frac{dDu}{d\mu})\,d\mu\ .$$

Unfortunately this functional is not lower semicontinuous with respect to the weak convergence in BV. The following example shows it.

Let $f\in C^\infty(\mathbb{R})$ with $f(1) = f(-1) = \varepsilon$, $f(0) = 1, \varepsilon \leq f \leq 1$ and

$$\int_{-1}^{1} f(u)du = 2\varepsilon + c < 2$$

with $c > 0$. Consider the functional

$$\mathcal{F}[u] = \int_{-1}^{1} f(u)|Du|$$

For a sequence $\{u_k\}$ of smooth non-decreasing functions with $u_k(-1) = -1$ $u_k(1) = 1$, which weakly converge in $BV(-1,1)$ to the function $u(x) = \operatorname{sign} x$, since $u(0)$ = average of $u = 0$, we have

$$\bar{\mathcal{F}}[u] = 2f(u(0)) = 2$$

$$\mathcal{F}[u_k] = \int_{-1}^{1} f(u_k)\,u_k'\,dx = \int_{-1}^{1} f(u) = 2\varepsilon + c < 2\ .$$

In this situation we had defined $u(0) = -1$, all would turn put well, but only for this special f and special $\{u_k\}$. In fact, it is not difficult to verify that any pointwise definition of u(0) does not allow to obtain semicontinuity for the general functional.

All the references can be found at the end of part II published in this issue.

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