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ON THE REGULARITY OF WEAK SOLUTIONS TO NONLINEAR ELLIPTIC SYSTEMS VIA LIOUVILLE'S TYPE PROPERTY

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Abstract: Let u be a weak solution with bounded gradient of a nonlinear elliptic system. In the present paper it is proved that the first derivatives of u are Hölder-continuous if the system satisfies a Liouville's type condition. This condition, roughly speaking, means that every solution defined on the whole R^n to the system and with bounded gradient is a polynomial of at most first degree.

Key words: Regularity, weak solution, nonlinear elliptic system, Liouville's property, Sobolev space.

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§ 1. Introduction. Let $\Omega \subset R^n$, $n \geqq 2$, be a bounded domain and let us consider a nonlinear elliptic system

(1.1) $$-\frac{\partial}{\partial x_i}[a_i^r(x,u,\nabla u)] + a^r(x,u,\nabla u) = -\frac{\partial f_i^r}{\partial x_i} + f^r,$$

$r = 1,2,\dots,m$, where $u \in [W^{1,\infty}(\Omega)]^m$, ∇u is the set of the derivatives $\frac{\partial u_s}{\partial x_j}$, $a_i^r(x,\xi,\eta)$, $\frac{\partial a_i^r}{\partial x_\ell}$, $\frac{\partial a_i^r}{\partial \xi_s}$, $\frac{\partial a_i^r}{\partial \eta_j^s}$ are continuous functions on $\Omega \times R^m \times R^{nm}$, $f_i^r \in W^{1,p}(\Omega)$, $f^r \in W^{1,\frac{p}{2}}(\Omega)$, $p > n$,

(1.2) $$\frac{\partial a_i^r}{\partial \eta_j^s}(x,\xi,\eta)\eta_i^r \eta_j^s > 0 \text{ for } \eta \neq 0,$$

and the summation convention is used.

Here $W^{k,p}(\Omega)$ denotes, as usual, the Sobolev space of

$L^p(\Omega)$ functions whose derivatives up to order k are also $L^p(\Omega)$ functions.

We say that (1.1).(1.2) is a regular system (R) if a weak solution u belongs to the space $[C^{1,\alpha}(\Omega)]^m$, where, of course, $C^{1,\alpha}(\Omega)$ is the space of continuously differentiable functions in Ω whose derivatives are locally α-Hölder continuous,

The history of the regularity problem is described in the book by O.A. Ladyženskaja, N.N. Ural'ceva [1], in the paper by Ch.B. Morrey [2] abd elsewhere. It is well known from the result of E. De Giorgi [3] that, for m = 1, the single equation (1.1),(1.2) is regular. By virtue of a counter example of J. Nečas [4], there exist systems (1.1),(1.2) which are not regular for $n \geqq 5$; this question is still open for n = = 3,4. Sufficient conditions for the regularity are also of interest, see M. Giaquinta [5], J. Nečas [6]. Since the examples of the regularity are $[W^{1,\infty}(R^n)]^m$ solutions to a system

$$(1.3)\qquad -\frac{\partial}{\partial x_i}[a_i^r(\nabla u)] = 0$$

of the type

$$(1.4)\qquad |x - x_0|\, g\left(\frac{x-x_0}{|x-x_0|}\right),$$

and in virtue of a trivial fact that a $C^1(R^n)$ vector function of the type (1.4) is a polynomial of at most first degree, we see that the regularity implies weak Liouville's property: we say that the system (1.1),(1.2) has weak Liouville's property (WL), if for every $x^0 \in \Omega$, $\xi \in R^m$, every function v with a bounded gradient of the type (1.4), solving in R^n the system

$$(1.5) \qquad -\frac{\partial}{\partial x_i}[a_i^r(x^0, \xi, \nabla v)] = 0,$$

is a polynomial (more exactly, a vector of polynomials) of at most first degree. We speak about <u>Liouville's property</u> (L) if the same is true without supposing (1.4).

We prove in this paper by the "partial regularity" method, see Ch. B. Morrey [7], E. Giusti, M. Miranda [8], E. Giusti [9], M. Giaquinta [5], that (L) $\Longrightarrow$ (R). In this connection 3! relations can be thought of between (R), (WL), (L) (some are trivial), especially (WL) $\overset{?}{\Longrightarrow}$ (R), (R) $\overset{?}{\Longrightarrow}$ (L).

Considering the solutions to (1.3) in the form (1.4), we can get, see J. Nečas [6], that, for $m = 1$, $n \geqq 2$, we have (WL). Because there is still some hope that for $n = 3,4$ we get (R) for the systems (1.1),(1.2) it is not unthinkable that we have the property (L) for $n = 3,4$, which would be a way how to prove this conjecture.

Clearly there are many other interesting questions, as, for example, how to avoid the condition $u \in [W^{1,\infty}(\Omega)]^m$; this seems to be possible via some growth conditions.

We also prove (in § 3 of this paper) an easy result that for the systems (1.1).(1.2) and for $n = 2$, the property (L) is satisfied. So we get once more the known result that for $n = 2$ we have (R).

§ 2. <u>Lemmas</u>. Let us first introduce some notation:

put $$u_{x_0,R} = \frac{1}{\text{mes } B_R(x_0)} \int_{B_R(x_0)} u(x)dx,$$

where $B_R(x_0)$ is the ball with the center x_0 and the radius R and

$$U(x_0,R) = R^{-m} \int_{B_R(x_0)} |u(x) - u_{x_0,R}|^2\, dx.$$

Let us mention the result of S. Campanato [10]: if

$$u \in [W^{1,2}_{loc}(B(0,1)) \cap L^2(B(0,1))]^m$$

is a weak solution to the equation with constant coefficients

$$\int_{B(0,1)} b^{hk}_{ij} D_i u_h D_j \psi_k \, dx = 0, \qquad \forall \psi \in [\mathcal{D}(B(0,1))]^m,$$

then for every $0 < \rho < 1$ we have

(*) $U(0,\rho) \leq c\,\rho^2\, U(0,1)$, where c depends on $\max |b^{hk}_{ij}|$

and on the constant α of ellipticity:

$$b^{hk}_{ij} \eta^h_i \eta^k_j \geq \alpha |\eta|^2 .$$

First we get a modification of the main lemma from [8], [9],[5].

<u>Lemma 2.1.</u> Let $v \in [L^\infty(\Omega)]^N \cap [W^{1,2}(\Omega)]^N$ be a weak solution to the system

$$(2.1) \quad \int_\Omega [A^{hk}_{ij}(x,v) D_i v_h D_j \varphi_k + A^{hk}_i(x,v) D_i v_h \varphi_k] dx =$$
$$= \int_\Omega [g^k_j D_j \varphi_k + g^k \varphi_k] dx,$$

where $\Omega \subset R^n$ is a bounded domain, $A^{hk}_{ij}(x,\xi)$, $A^{hk}_i(x,\xi)$ are continuous functions in $\bar{\Omega} \times R^N$, $g^h_j \in L_p(\Omega)$, $g^k \in L_{\frac{p}{2}}(\Omega)$, $p > n$,

$$(2.2) \qquad A^{hk}_{ij}(x,\xi)\, \eta^h_i \eta^k_j > 0 \text{ for } \eta \neq 0.$$

If $x_0 \in \Omega$ and $R \leq \mathrm{dist}(x_0, \partial\Omega)$ we put $v = v^* + w$, where $w \in H^1_0(B(x_0,R))$ is a solution to

$$(2.2') \quad \int_{B(x_0,R)} [A^{hk}_{ij}(x,v) D_i w_h D_j \varphi_k + A^{hk}_i(x,v) D_i w_h \varphi_k] dx =$$

$$= \int_{B(x_0,R)} [g_j^k D_h \varphi_k + g^k \varphi_k] dx.$$

Then for every τ, $0 < \tau < 1$, there exist $\varepsilon_0 = \varepsilon_0(\tau, |v|_\infty)$, $R_0 = R_0(\tau, |v|_\infty)$ such that if $R \leq \min(R_0, \mathrm{dist}(x_0, \partial\Omega))$ and if

$$V^*(x_0,R) < \varepsilon_0^2 \tag{2.3}$$

then

$$V^*(x_0, \tau R) \leq 2c\, \tau^2\, V^*(x_0,R), \tag{2.4}$$

where the constant c is from (*).

<u>Proof</u>. Let us suppose the contrary. Then $\exists \tau$, $x_\nu \in \Omega$, $\varepsilon_\nu \longrightarrow 0$, $R_\nu \longrightarrow 0$, $v^\nu \in [H^1(\Omega)]^N$, $|v^\nu|_{L_\infty} \leq |v|_{L_\infty}$, such that $V^{*(\nu)}(x_\nu, R_\nu) = \varepsilon_\nu^2$, $V^{*(\nu)}(x_\nu, \tau R_\nu) > 2c\,\tau^2\,\varepsilon_\nu^2$. Put $x = x_\nu + R_\nu y$, $s^\nu(y) = \varepsilon_\nu^{-1}[v^{*\nu}(x_\nu + R_\nu y) - v^{*\nu}_{x_\nu, R_\nu}]$. We have $\int_{B(0,1)} |s^\nu(y)|^2 dy = S^\nu(0,1) = 1$,

$$S^\nu(0,\tau) > 2c\,\tau^2. \tag{2.5}$$

Put further $t^\nu(y) = \omega^\nu(x_\nu + R_\nu y)$ $(v^\nu = v^{*\nu} + \omega^\nu)$. Then we can suppose $x_\nu \longrightarrow x_0 \in \bar\Omega$, $s^\nu \rightharpoonup s$ in $L_2(B(0,1))$, $\varepsilon_\nu s^\nu(y) \longrightarrow 0$ almost everywhere in $B(0,1)$. We have

$$v^\nu(x_\nu + R_\nu y) = s^\nu(y)\varepsilon_\nu + v^{*\nu}_{x_\nu,R_\nu} + t^\nu(y). \tag{2.6}$$

Since

$$\int_{B(x_\nu,R_\nu)} |\omega^\nu|^2 dx \leq c_1 R_\nu^2 \int_{B(x_\nu,R_\nu)} D_i\omega_h^\nu D_i\omega_h^\nu\, dx, \tag{2.7}$$

we first get that (2.2) is uniquely solvable for R_ν small enough. We further get from (2.7) and (2.2) that

$$\int_{B(0,1)} |t^\nu(y)|^2 dy \leq c_2 R^{2(1-\frac{n}{p})}, \tag{2.8}$$

so we can also suppose that $t^{\nu}(y) \longrightarrow 0$ almost everywhere. Hence from (2.6) it follows that we can suppose $v^{*\nu}_{x_\nu, R_\nu} \longrightarrow \xi \in R^N$ and therefore

$$A^{hk}_{ij}(x_\nu + R_\nu y, s^{\nu}(y)\varepsilon_\nu + v^{*\nu}_{x_\nu, R_\nu} + t^{\nu}(y)) \longrightarrow A^{hk}_{ij}(x^0, \xi)$$

almost everywhere in $B(0,1)$. Hence we get that $s^{\nu} \longrightarrow s$ in $[W^{1,2}_{loc}(B(0,1))]^N$ and that

$$(2.9) \qquad \int_{B(0,1)} A^{hk}_{ij}(x^0, \xi) D_i s_h D_j \psi_k \, dy = 0 \qquad \forall \psi \in [\mathcal{D}(B(0,1))]^N.$$

Thus we have

$$(2.10) \qquad S(0,\tau) \leq c\,\tau^2\, S(0,1) \leq c\,\tau^2,$$

which is a contradiction with

$$(2.11) \qquad S(0,\tau) > 2c\,\tau^2$$

obtained from (2.5).

<u>Lemma 2.2.</u> Under the conditions of Lemma 2.1, for every point $x_0 \in \Omega$ such that $V^*(x_0,R) < \varepsilon_0^2$, there exists a $B(x_0,R_1) \subset \Omega$ such that $v \in C^{\alpha}(\overline{B(x_0,R_1)})$ with $\alpha = \min(\frac{1}{2}, 1 - \frac{n}{p})$.

<u>Proof.</u> We get by a standard argument that if $\delta > 0$ is small enough, $|\bar{x} - x_0| < \delta$, and $R_{\bar{x}} = R - |\bar{x} - x_0|$, then $V^*(\bar{x}, R_{\bar{x}}) < \varepsilon_0^2$. If $v = v^* + \omega$ in $B(\bar{x}, R_{\bar{x}})$, we first have

$$(2.12) \qquad \int_{B(\bar{x},R_{\bar{x}})} |\omega|^2 dx \leq c_1 R_{\bar{x}}^2 \int_{B(\bar{x},R_{\bar{x}})} |D\omega|^2 dx \leq$$
$$\leq c_2 R_{\bar{x}}^2 R_{\bar{x}}^{n(1-\frac{2}{p})} [\Sigma [\int_B |f_i^r|^p dx]^{\frac{2}{p}} + \Sigma (\int_B |f^r|^{\frac{p}{2}} dx)^{\frac{4}{p}}] \leq c_3 R_{\bar{x}}^{2+n-\frac{2n}{p}} .$$

Thus

$$(2.13)\qquad V(\bar{x},\tau R_{\bar{x}}) \leq 2V^*(\bar{x},\tau R_{\bar{x}}) + 2\Omega(\bar{x},\tau R_{\bar{x}}) \leq$$

$$\leq 4c\,\tau^2 V^*(\bar{x},R_{\bar{x}}) + 2c_3 R_{\bar{x}}^{2(1-\frac{n}{p})}\ \tau^{-n} \leq$$

$$\leq 8c\,\tau^2 V(\bar{x},R_{\bar{x}}) + 8c\,\tau^2 c_3 R_{\bar{x}}^{2(1-\frac{n}{p})} + 2c_3\tau^{-n} R_{\bar{x}}^{2(1-\frac{n}{p})}.$$

Choose $\tau \in (0,1)$ such that $8c\tau = \varkappa \leq 1$ and small enough. We get from (2.13) that

$$(2.14)\qquad V(\bar{x},\tau R_{\bar{x}}) \leq 8c\,\tau^2\ V(\bar{x},R_{\bar{x}}) + c_4\ R_{\bar{x}}^{2(1-\frac{n}{p})}.$$

For k being a positive integer, we get from (2.14) that

$$(2.15)\qquad V(\bar{x},\tau^k R_{\bar{x}}) \leq \varkappa^k\ V(\bar{x},R_{\bar{x}}) +$$

$$+ R_{\bar{x}}^{2(1-\frac{n}{p})}\ c_4\ \frac{(\varkappa\tau)^k + \tau^{k2(1-\frac{n}{p})}}{|\varkappa\tau - \tau^{2(1-\frac{n}{p})}|}\ .$$

If $0 < \rho < R - \delta$ and if we choose k such that $\tau^{k+1}R_{\bar{x}} < \rho \leq \tau^k R_{\bar{x}}$, we get $\tau^n V(\bar{x},\rho) \leq \left(\frac{\rho}{\tau^k R_{\bar{x}}}\right)^n V(\bar{x},\rho) \leq V(\bar{x},\tau^k R_{\bar{x}}) \leq$ $\leq \frac{\rho}{R_{\bar{x}}\tau}\ V(\bar{x},R_{\bar{x}}) + c_4 R_{\bar{x}}^{2(1-\frac{n}{p})}\ \frac{\frac{\rho}{R_{\bar{x}}\tau} + \left(\frac{\rho}{R_{\bar{x}}\tau}\right)^{2(1-\frac{n}{p})}}{|\varkappa\tau - \tau^{2(1-\frac{n}{p})}|}$, and using [10], we get the result, q.e.d.

§ 3. Main results

Theorem 1. Let $u \in [W^{1,\infty}(\Omega)]^m$ be a weak solution to (1.1) and let the conditions on a_i^r, a^r, f_i^r, f^r, Ω, mentioned in § 1, be fulfilled. Let the system (1.1) satisfy the Liouville's property, i.e., for $\forall x^\circ \in \Omega$ and $\forall \xi \in R^m$ the only solution to (1.5) defined in the whole R^n and possessing a bounded gradient is a polynomial of at most first degree.

Then $u \in [C^{1,\alpha}(\Omega)]^m$, $\alpha = \min(\frac{1}{2}, 1 - \frac{n}{p})$.

<u>Proof.</u> Let $x^0 \in \Omega$. Put $u_R(y) = \frac{1}{R}[u(x^0 + Ry) - u(x^0)]$, $x^0 + Ry = x$. If O is the image of Ω we have

$$(3.1) \quad \int_O [a_i^r(x^0+Ry,\ Ru_R(y)+u(x^0),\ \nabla_y u_R(y)) \frac{\partial \psi_r(y)}{\partial y_i} +$$

$$+ a^r(x^0+Ry,\ Ru_R(y) + u(x^0),\ \nabla_y u_R(y)) R \psi_r(y)]\, dy =$$

$$= \int_O [f_i^r(x^0+Ry) \frac{\partial \psi_r}{\partial y_i}(y) + f^r(x^0+Ry) R \psi_r(y)] dy.$$

Let $B(0,a) \subset O$. We get in a standard way that

$$(3.2) \quad \int_{B(0,a)} |D^2 u_R(y)|^2\, dy \leqq c(a).$$

Hence we can choose $R_k \to 0$ in such a way, that $u_{R_k} \to p$ in $[W^{1,2}(B(0,a))]^m$ $\forall a > 0$. Thus $p \in [W^{1,\infty}(R^n)]^m$ and it is a weak solution to

$$(3.3) \quad \int_{R^n} a_i^r(x^0, u(x^0),\ \nabla_y p) \frac{\partial \psi_r}{\partial y_i}\, dy = 0$$
$$\forall \psi \in [\mathcal{D}(R^n)]^m.$$

Therefore, by assumption, p is a polynomial of at most first degree. So we have

$$(3.3') \quad 0 \leftarrow \int_{B(0,1)} |Du_{R_k}(y) - Dp|^2\, dy = R_k^{-n} \int_{B(x^0, R_k)} |Du(x) - Dp|^2 dx.$$

If $'$ is the $\frac{\partial}{\partial x_t}$ derivative we get from (1.1) the equation in variations

$$(3.4) \int_\Omega [\frac{\partial a_i^r}{\partial \frac{\partial u_s}{\partial x_j}} \frac{\partial u_s'}{\partial x_j} \frac{\partial \varphi_r}{\partial x_i} + \frac{\partial a_i^r}{\partial u_s} u_s' \frac{\partial \varphi_r}{\partial x_i} + \frac{\partial a_i^r}{\partial x_t} \frac{\partial \varphi_r}{\partial x_i} +$$

$$+ \frac{\partial a^{r}}{\partial \frac{\partial u_s}{\partial x_j}} \frac{\partial u_s}{\partial x_j} \varphi_r + \frac{\partial a^{r}}{\partial u_s} u_s' \varphi_r + \frac{\partial a^{r}}{\partial x_t} \varphi_r] dx =$$

$$= \int_{\Omega} [f_i^{r'} \frac{\partial \varphi_r}{\partial x_i} + f^{r'} \varphi_r] dx .$$

Writing (3.4) for every $\frac{\partial}{\partial x_t}$, $t = 1,2,\dots,n$, removing the terms $\frac{\partial a_i^r}{\partial u_s} u_s' \frac{\partial \varphi_r}{\partial x_i}$, $\frac{\partial a_i^r}{\partial x_t} \frac{\partial \varphi_r}{\partial x_i}$, $\frac{\partial a^r}{\partial u_s} u_s' \varphi_r$, $\frac{\partial a^r}{\partial x_t} \varphi_r$ to the right-hand side of (3.4), and denoting by $v_{\varkappa}$ the derivatives $\frac{\partial u_s}{\partial x_t}$, we get, with $\frac{\partial a_i^r}{\partial \frac{\partial u_s}{\partial x_j}}(x,u(x),v) \equiv b_{ij}^{rs}(x,v)$ (and the same with a_i^r), a system of the type (2.1). The result follows from Lemmas 2.1, 2.2 and from (3.3′), because, in decomposing $v = v^* + \omega$ on $B(x^0,R)$ as in Lemma 2.1, we have $\Omega(x^0,R) \longrightarrow 0$ for $R \longrightarrow 0$, as above, so $V^*(x^0,R) \longrightarrow 0$, q.e.d.

Theorem 2. Let us consider the system (1.1),(1.2). Let n be the dimension of the space, n = 2. Then (L) is satisfied.

Proof. Let $v \in [W^{1,\infty}(R^2)]^m$ be a weak solution to the equation

$$(3.5) \qquad \int_{R^2} a_i^r(x^0, \xi, \nabla v) \frac{\partial \psi_r}{\partial y_i} dy = 0.$$

Let $T > 0$ and let $\eta \in \mathcal{D}(B(0,2T))$, $0 \leq \eta \leq 1$, $\eta = 1$ in $B(0,T)$, $|D_i \eta| \leq \frac{c_1}{T}$. We get the equation in variations

$$(3.6) \qquad \int_{R^2} \frac{\partial a_i^r}{\partial \frac{\partial v_s}{\partial x_j}}(x^0, \xi, \nabla v) \frac{\partial v_s'}{\partial y_j} \cdot \frac{\partial \psi_r}{\partial y_i} dy = 0.$$

Putting $\psi_r = v_r' \eta^2$, we get from (3.6), using the boundedness of the gradient, that

(3.7) $$\int_{B(0,2T)} |Dv'|^2 \eta^2 \, dy \leq c_2 .$$

Hence $\int_{R^2} |Dv'|^2 \, dy < \infty$. But there exists $\psi^n \in [\mathcal{D}(R^2)]^m$ such that $D\psi^n \longrightarrow Dv'$ in $[L^2(R^2)]^{2m}$ (and there exists $\Lambda^n \in R^m$ such that $\Lambda^n + \psi^n \longrightarrow v'$ in $[L^2_{loc}(R^n)]^m$. Hence

$$\int_{R^2} \frac{\partial a_i^r}{\partial \frac{\partial v_s}{\partial x_j}} (x^{\bullet}, \xi, \nabla v) \frac{\partial v'_s}{\partial y_j} \frac{\partial v'_r}{\partial y_i} \, dy = 0$$

and thus v is a polynomial of at most first degree.

R e f e r e n c e s

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