COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
20, 1 (1979)

ON EXTENSIONS OF UNIFORMLY CONTINUOUS BANACH-SPACE-VALUED MAPPINGS

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Abstract: Suppose that for any uniform covering $\mathcal{X}$ of a uniform space X there exists a partition of unity subordinated to $\mathcal{X}$ which is strongly equiuniformly continuous (i.e. such a partition of unity $\{f_\alpha \mid \alpha \in I\}$ that $\{\sum_{\alpha \in J} f_\alpha \mid J \subset I\}$ is equiuniformly continuous). Then any uniformly continuous mapping from a subspace of X into a closed ball of a Banach space can be extended over X.

Key words: Partition of unity, Banach-space-valued uniformly continuous mapping.

AMS: 54C20, 54E15

Let $\mathcal{X}$ be a uniform covering of a space X. We say that a family $\{f_\alpha \mid \alpha \in I\}$ of non-negative real-valued functions is a strongly equiuniformly continuous partition of unity (abbr. SECPU) subordinated to $\mathcal{X}$ if

1. for any $x \in X$, $\sum_{\alpha \in I} f_\alpha (x) = 1$ and the covering $\{\mathrm{coz}\, f_\alpha \mid \alpha \in I\}$ refines $\mathcal{X}$.
2. $\{\sum_{\alpha \in J} f_\alpha \mid J \subset I\}$ is equiuniformly continuous ([I],p. 62).

The notion of SECPU has been defined by Z. Frolík in [F] (under the name of l_1-continuous partition of unity).

Denote by $\mathcal{S}$ the class of all uniform spaces determined by the property that, for any $X \in \mathcal{S}$ and any $\mathcal{X} \in X$, there is a SECPU subordinated to $\mathcal{X}$. We can see easily that $\mathcal{S}$ i-a reflective subcategory of uniform spaces containing "fine" spaces (e.g. locally fine and metric fine) and "coarse" spaces (e.g. the spaces with finite-dimensional base). Recall also that $\mathcal{S}$ does not contain all uniform spaces as was established by M. Zahradník [Z] and will follow from our note, too.

It should be remarked that the theorem we are about to prove has been proved for spaces with finite-dimensional base by G. Vidossich [V]. We have adopted some of his technique.

<u>Theorem</u>: Suppose that B is a Banach space and X a uniform space belonging to $\mathcal{S}$. Let $U^*(X,B)$ stand for the Banach space of all bounded uniformly continuous mappings from X into B. Then, for any $Y \subset X$, there is a continuous norm-preserving extender $e: U^*(Y,B) \longrightarrow U^*(X,B)$.

Proof: We show firstly that any bounded uniform $f: Y \longrightarrow B$ can be extended over X in such a way that the resulting mapping is bounded, too. We may and shall assume that f ranges in the 1-ball $B[0,1]$, which is centered in 0. Let $\mathcal{B}_n$ be a uniform covering of $B[0,1]$ consisting of all $\frac{1}{2^n}$ -balls. Take uniform coverings $\mathcal{X}_n = \{X^n_\alpha \mid \alpha \in I\}$ of X such that $\mathcal{X}_n$ restricted to Y refines $f^{-1}(\mathcal{B}_n)$ and take a SECPU, $\{f^n_\alpha \mid \alpha \in I\}$, subordinated to $\mathcal{X}_n$. Choose points $y^n_\alpha \in B[0,1]$ such that $y^n_\alpha = 0$ whenever $X^n_\alpha \cap Y = \emptyset$, otherwise set $y^n_\alpha \in f(X^n_\alpha \cap Y)$ arbitrarily. Suppose we have done that for all $n \in N$. Now, put $g^n(x) = \sum_{\alpha \in I} f^n_\alpha(x) \cdot y^n_\alpha$. Observe that the definition of g^n is correct because, for any $x \in X$, the series $\sum_{\alpha \in I} f^n_\alpha(x) \cdot y^n_\alpha$ is Cauchy in $B[0,1]$ and so it has a limit $g^n(x)$. We shall prove that any g^n is uniformly continuous and moreover, g^n tend uniformly to f on Y. As always, $\| g^n(x) - g^n(y) \| =$
$= \| \sum_{\alpha \in I} f^n_\alpha(x) y^n_\alpha - \sum_{\alpha \in I} f^n_\alpha(y) \cdot y^n_\alpha \| \leq \sum_{\alpha \in I} | f^n(x) - f^n_\alpha(y) |$ and

therefore g^n is uniformly continuous in view of the fact that $\{f^n_\alpha \mid \alpha \in I\}$ is SECPU. Further, if $x \in Y$ then $\| f(x) - g^n(x) \| =$ $= \| \sum_{\alpha \in I} f^n_\alpha(x) \cdot f(x) - \sum_{\alpha \in I} f^n_\alpha(x) \cdot y^n_\alpha \| \leq \sum_{\alpha \in I} | f^n_\alpha(x) | \; \| f(x) -$ $- y^n_\alpha \| \leq \frac{1}{2^{n-1}}$. Therefore g^n converge uniformly to f on Y. To conclude the first part of the proof, define mappings $h^n: X \longrightarrow$ $\longrightarrow B$ as follows: $h^n(x) = g^{n+1}(x) - g^n(x)$ provided $\| g^{n+1}(x) -$ $- g^n(x) \| \leq \frac{1}{2^{n-1}}$, otherwise set $h^n(x) = \frac{1}{2^{n-1}} \frac{g^{n+1}(x) - g^n(x)}{\| g^{n+1}(x) - g^n(x) \|}$. It is easy to see that all h^n are uniformly continuous, $\| h^n \| \leq \frac{1}{2^{n-1}}$ and $h^n = g^{n+1} - g^n$ on Y. Therefore $\sum_{n=1}^{\infty} h^n$ is a uniformly continuous mapping and $\sum_{n=1}^{\infty} h^n = f - g^1$ on Y.

To conclude the proof, we will use the Bartle-Graves theorem the same way as in [Lu]. As the restriction operator r: $: U^*(X,B) \longrightarrow U^*(Y,B)$ is a continuous linear surjection we have a cpntinuous extender $\tilde{e}: U^*(Y,B) \longrightarrow U^*(X,B)$ which can be altered to a norm-preserving one by putting $e(f) = \tilde{e}(f)$ as long as $\| \tilde{e}(f)(x) \| \leq \| f \|$, $e(f)(x) = \frac{\tilde{e}(f)(x)}{\| e(f)(x) \|}$ whenever $\| \tilde{e}(f)(x) \| > \| f \|$. The mapping $e(f)$ is uniformly continuous.

Let us make two comments in conclusion. Firstly, note that we obtain as a corollary another proof of the following famous theorem (see [Lu]): If Y is a closed subspace of a collectionwise normal topological space X and B is a Banach space then there is a continuous norm-preserving extender e: $: C^*(Y,B) \longrightarrow C^*(X,B)$. Indeed, the fine uniformity on Y is the restriction of the fine uniformity on X because X is collectionwise normal.

Secondly, it follows immediately that $l_\infty(N)$ does not belong to $\mathcal{S}$. It can be derived from our Theorem and the fact that there is a uniform mapping from a subspace of $l_\infty(N)$

into a closed ball in a Banach space which does not have an extension over X (see [Li], also [Vi] for relevant comments). Of course, this observation does not mean any discovery for the time being as M. Zahradník showed recently that even any Banach infinitely dimensional space does not belong to $\mathcal{S}$, [Z]. As Zahradník's procedure seems rather elaborate it would be nice and should be possible to establish the full Zahradník's result by our method. We have not been able to do so.

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(Oblatum 25.10.1978)