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GAUSSIAN MEASURES AND COVERING THEOREMS

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Abstract: It is shown that Vitali type covering theorem does not hold for (centered) families of balls in Hilbert spaces and Gaussian measures.

Key words: Vitali type covering theorem, Gaussian measures in Hilbert spaces.

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Vitali type covering theorems in finite dimensional Banach spaces hold not only for the Lebesgue measure but also (under some regularity assumptions on the considered covers) for arbitrary (locally finite) measures (see [B], [M], [F, p. 147-150], [T] for more details). If we drop the assumption of finite dimensionality the situation becomes different. By a result of Roy O. Davies [D] there exist distinct probability measures on a metric space which agree on all balls. This particular behaviour is not possible in the case of Hilbert spaces. Indeed, if μ, ν are positive finite measures on a Hilbert space H which agree on balls then $\int \exp(\frac{1}{2}\|x+y\|^2)\,d\mu(x) = \int \exp(\frac{1}{2}\|x+y\|^2)\,d\nu(x)$ for every $y \in H$, consequently $\int \exp(i(x,y))\,\exp(\frac{1}{2}(x,x))\,d\mu(x) = \int \exp(i(x,y))\,\exp(\frac{1}{2}(x,x))\,d\nu(x)$. This implies that the Fourier transform of $\exp(\frac{1}{2}(x,x))\mu$ and $\exp(\frac{1}{2}(x,x))\nu$ coincide, hence $\mu = \nu$.

However, in this note we prove that Vitali type theorem does not hold (even in a restricted sense, i.e. for the Vitali system $\mathcal{V}_0$ of [T]) for Gaussian measures in infinitely dimensional separable Hilbert spaces.

Recall that a measure γ in $\mathbb{R}^n$ is called Gaussian if there is a positive quadratic form $A(x,y)$ on $\mathbb{R}^n$ such that $\gamma(M) = \frac{1}{N}\int_M \exp(-A(x,x))\,d\mathcal{L}^n x$ (where $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$); the normalizing factor N is chosen so that $\gamma(\mathbb{R}^n) = 1$. A measure γ on a separable Hilbert space is called Gaussian if $\pi[\gamma]$ is Gaussian whenever π is a continuous linear map of H onto $\mathbb{R}^n$.

We shall construct our example in $H = \ell_2$; the closed ball in H with the center x and radius r will be denoted $B(x,r)$ and the closed ball in $\mathbb{R}^n$ (considered here with the ℓ_2^n-norm) with the center in x and radius r will be denoted $B_n(x,r)$.

Lemma 1. There is a sequence (a_n) of positive real numbers with $\sum a_n < \infty$ such that $\mathcal{L}^n(\bigcup_{t\in T} B_n(x_t,r)) \leq a_n \mathcal{L}^n(C)$ whenever C is an open cube in $\mathbb{R}^n$ (with its sides parallel to the coordinate axes), $r > 0$, $B_n(x_t,r) \subset C$ for every $t \in T$ and the family $\{B_n(x_t,r), t \in T\}$ is disjoint.

Proof. Let (a_n) be the sequence of packing densities of balls in $\mathbb{R}^n$ (see [R, p. 24] for the definitions). The convergence of $\sum a_n$ follows from [R, Theorem 7.1] and Daniels's asymptotic formula [R, p. 90, formula (1)]. The inequality $\mathcal{L}^n(\bigcup_{t\in T} B_n(x_t,r)) \leq a_n \mathcal{L}^n(C)$ follows from [R, Theorem 1.5].

Lemma 2. Let (a_n) be the sequence from the preceding Lemma and let γ be a Gaussian measure in $\mathbb{R}^n$. Then there is $\delta > 0$ such that $\gamma(\bigcup_{t\in T} B_n(x_t,r)) \leq 5\, a_n$ whenever $0 < r < \delta$

and the family $\{B_n(x_t,r); t\in T\}$ to disjoint.

Proof. Let C_o be a cube in $\mathbb{R}^n$ such that $\gamma(\mathbb{R}^n - C_o)\leq$ $\leq a_n$. There is a partition of C_o into cubes C_i ($i = 1,2,\ldots$ $\ldots,N$) and positive numbers z_i such that $z_i\,\mathcal{L}^n(M)\leq\gamma(M)\leq$ $\leq 2z_i\,\mathcal{L}^n(M)$ whenever $M\subset C_i$ (consider any partition of C_o into sufficiently small cubes). Choose $\delta>0$ such that $1-(1-2\delta)^n\leq a_n$. Then, using Lemma 1, we obtain

$$\gamma(\bigcup_{t\in T} B_n(x_t,r)) \leq \sum_{i=1}^{N} 2z_i\,[\mathcal{L}^n(\bigcup_{B_n(x_t,r)\subset C_i} B_n(x_t,r)) + (1-(1-2\delta)^n)\,\mathcal{L}^n(C_i)] + a_n \leq \sum_{i=1}^{N} 4a_n z_i\,\mathcal{L}^n(C_i) + a_n \leq$$
$$\leq 4a_n\,\gamma(C_o) + a_n \leq 5a_n.$$

<u>Theorem</u>. There exist a Gaussian measure γ in ℓ_2, a subset M of ℓ_2 and a subset S of $(0,+\infty)$ such that

(i) M is γ-measurable and $\gamma(M)>0$

(ii) $S\cap(0,h)\neq\emptyset$ for each $h>0$

(iii) $\lim_{h\to 0+}[\sup\{\gamma(\bigcup\{B, B\in\mathcal{S}\}$; $\mathcal{S}$ is a disjoint family of balls in ℓ_2 with centers in M and radii belonging to $S\cap(0,h)\}] = 0$.

Proof. Let (a_n) be the sequence from Lemma 1. We shall construct sequences R_i, r_i, ε_i of real numbers and sequences γ_i of Gaussian measures in $\mathbb{R}^i$ and ν_i of Gaussian measures in $\mathbb{R}$ such that

(1) $0<\varepsilon_i<r_i<R_i\leq 1/i$

(2) $R_i\leq 2^{-i}\min\{\varepsilon_j, 1\leq j<i\}$ for $i = 2,3,\ldots$

(3) $\nu_i(B_1(0,R_i))\geq 1-2^{-i-1}$

(4) $\gamma_i = \prod_{j=1}^{i}\nu_j$

(5) $\gamma_i(\bigcup_{t\in T} B_i(x_t,r_i))\leq 5\,a_i$ whenever the family $\{B_i(x_t,r_i); t\in T\}$ is disjoint

(6) $\gamma_i(B_i(x,r_i+\varepsilon_i))\leq 2\gamma_i(B_i(x;r_i))$ whenever $x\in B_i(0,\sum_{k=1}^{i} R_k)$.

For $i = 1$ we can put $R_1 = 1$, choose a Gaussian measure $\nu_1 = \gamma_1$ such that (3) holds, then choose $r_1 < R_1$ fulfilling (5) according to the preceding Lemma; the condition (6) clearly holds for sufficiently small positive $\varepsilon_1 < r_1$.

The induction step is also easy. We may first choose $R_i \leq 1/i$ such that (2) holds, then find a Gaussian measure ν_i fulfilling (3) and then choose $r_i < R_i$ according to Lemma 2; the condition (6) again holds for all sufficiently small $\varepsilon_i < r_i$.

Let $\varkappa_i : \ell_2 \longrightarrow \mathbb{R}$ be the i-th coordinate and let $\pi_i : \ell_2 \longrightarrow \mathbb{R}^i$ be the projection into the first i coordinates. From (1) and (3) we infer that there is a unique (necessarily Gaussian) measure γ on ℓ_2 such that $\int g(\pi_i z) d\gamma(z) = \int g(x) d\gamma_i(x)$ for $i = 1, \ldots$ and any bounded Borel function g on $\mathbb{R}^i$ (cf. [G]). Put $M = \bigcap_{i=1}^{\infty} \varkappa_i^{-1}(B_1(0,R_i))$; then (3) implies $\gamma(M) \geq 1/2$. Let S be the set of all numbers $r_i + \varepsilon_i$.

If $\mathcal{S}$ is a disjoint family of balls in ℓ_2 with radii in $S \cap (0, r_k + \varepsilon_k)$ put $\mathcal{S}_i = \{B(x,r) \in \mathcal{S}\ ;\ r = r_i + \varepsilon_i\}$ for $i = k+1, \ldots$.

Whenever $B(x, r_i + \varepsilon_i)$, $B(y, r_i + \varepsilon_i)$ belong to $\mathcal{S}_i$ and $x \neq y$ we have $4(r_i + \varepsilon_i)^2 < \| x - y \|^2 \leq \| \pi_i x - \pi_i y \|^2 + 4 \sum_{j>i} R_j^2 \leq \| \pi_i x - \pi_i y \|^2 + 4\varepsilon_i^2$ according to (2), hence the family $\{B_i(\pi_i x, r_i);\ B(x, r_i + \varepsilon_i) \in \mathcal{S}_i\}$ of balls in $\mathbb{R}^i$ is disjoint. Using (6) and (5) we obtain $\gamma(\bigcup \{B; B \in \mathcal{S}_i\}) \leq$
$\leq \sum \{\gamma(\pi_i^{-1}(B_i(\pi_i x, r_i + \varepsilon_i));\ B(x, r_i + \varepsilon_i) \in \mathcal{S}_i\} \leq$
$\leq \sum \{\gamma_i(B_i(\pi_i x, r_i + \varepsilon_i));\ B(x, r_i + \varepsilon_i) \in \mathcal{S}_i\} \leq$
$\leq 2 \sum \{\gamma_i(B_i(\pi_i x, r_i));\ B(x, r_i + \varepsilon_i) \in \mathcal{S}_i\} \leq 10\, a_i$.
Hence $\gamma(\bigcup \{B, B \in \mathcal{S}\}) \leq 10 \sum_{i>k} a_i$.

R e f e r e n c e s

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