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A NOTE ON THE EXISTENCE OF MORE THAN ONE SOLUTION FOR ASYMPTOTICALLY LINEAR EQUATIONS

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Abstract: Consider the nonlinear operator equation $Lu + N(u) = f$ with nonlinearity satisfying $P_0N(x_0) \longrightarrow 0$ as $\|x_0\| \longrightarrow \infty$ for x_0 in Ker L, P_0 being the projection onto Coker L. Under additional hypotheses we show that this equation has the property that for $\|P_0f\|$ sufficiently small, it has at least two solutions.

Key words: Fredholm, semilinear alternative problems, degree, Leray-Schauder degree, homotopy.

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Introduction. Consider the nonlinear operator equation

(A) $Lu + N(u) = f$

where L is a linear Fredholm map of index zero between Banach spaces X and Y and N is a compact uniformly bounded map of X into Y. Using the notation X_0 = Ker L, P_0 = projection onto Coker L, we decompose each x in X into $x_0 + x_1$ where $X = X_0 \oplus X_1$ and X_1 is some complement of X_0 in X. We assume

(H.1) Given $\varepsilon > 0$ and $k \geq 0$ there exists $\rho > 0$ such that if $\|x_1\| \leq k$ and $\|x_0\| \geq \rho$, $\|P_0N(x_0 + x_1)\| < \varepsilon$.

In addition, suppose Ker L is one-dimensional and

(H.2) For any M, there exists a number R_o such that if $\| x_1 \| \leq M$ and $\| x_o \| \geq R_o$ $P_oN(x_o + x_1)$ and $P_oN(-x_o + x_1)$ are of opposite signs.

Then the followin result is known:

Theorem. Assuming (H.1) and (H.2), the equation (A) has a solution for each f in the range of L. Furthermore there is a number c depending on P_1f, where $P_1 = I - P_o$ is the projection onto the range of L, such that for $\| P_of \| < c(P_1f)$ (A) has a solution.

Examples of boundary-value problems where essentially this abstract result is used can be found in references [1], [2], and [3] .

The generalization of this theorem to the case where dim Ker $L > 1$ is easily seen. Let $\{x_{oi}\}_{i=1,\dots,n}$ be a fixed basis of unit vectors spanning Ker L and let an arbitrary element of Ker L be denoted by $a \cdot x_o$ where $a = (a_1,\dots,a_n)$ $x_o = (x_{o1},\dots,x_{on})$ and $a \cdot x_o = a_1x_{o1} + \dots + a_nx_{on}$. Instead of (H.2) assume

(H.3) For any M there exists a number R_o such that $\| x_1 \| \leq M$ and $|a| \geq R_o$ imply $P_oN(a \cdot x_o + x_1) \neq 0$

and letting $\phi(a) = P_oN(a \cdot x_o)$ be regarded as a map of R^n into R^n, assume for $R \geq R_o$

(H.4) $\deg(\phi, 0, D_R^n) \neq 0$ where D_R^n is the ball of radius R in R^n and deg is the standard Brouwer degree.

Clearly for the case of a one-dimensional kernel, (H.3) and (H.4) are equivalent to (H.2). The result now reads as follows:

Theorem. Let L and N be as above with N satisfying (H.1), (H.3), and (H.4). Then for each f, there is a number $c(P_1f)$ such that for $\| P_of \| < c(P_1f)$, (A) has a solution.

A variant of this result has been proved and used by Mawhin in the study of periodic solutions of ordinary vector differential equations. (See [4] and [5]).

In this note we extend the results mentioned above by showing that for $\| P_of \|$ sufficiently small and $\neq 0$, (A) has in fact at least two solutions.

Section 1. Here we formally state and prove our main result.

Theorem 1. Suppose N satisfies (H.1), (H.3), and (H.4). Then for each f, there exists a number $c(P_1f)$ such that for $0 < \| P_of \| < c(P_1f)$, equation (A) has at least two solutions. Here $c(P_1f)$ is the same constant needed in the previously mentioned work.

To prove Theorem 1, using the standard method for semi-linear alternative problems, we rewrite (A) as

(1) $F(x_1,a) = 0$

where $F: X_1 \times R^n \longrightarrow X_1 \times R^n$ is given by

(2) $$F(x_1,a) = (x_1 + L^{-1}P_1 [N(a \cdot x_o + x_1) - f],\; P_oN(a \cdot x_o + x_1) - P_of)$$

Here P_1 is the projection onto $L(X_1)$ and $L: X_1 \longrightarrow L(X_1)$ has an inverse which we have denoted as L^{-1}.
Let $D_k = \{ (x_1,a): \| x_1 \| + |a| \leq k \}$ and let S_k be its boundary. Then we have

Lemma 1. There exist constants c and k such that if $\| P_0 f \| < c$, $\deg_{LS}(F,(0,0),D_k) \neq 0$, where $\deg_{LS}$ is the Leray-Schauder degree. Furthermore these constants depend on $P_1 f$.

Proof. Let

(3) $H(x_1,a,t) = (x_1 + tL^{-1}P_1 [N(a \cdot x_0 + x_1) - f]$,

$P_0 N(a \cdot x_0 + tx_1) - P_0 f)$

We claim that there exist constants, c, k such that if $\| P_0 f \| < c$, $H(x_1,a,t) \neq 0$ on S_k. This is easily seen since if the first component of H is zero, by (3),

(4) $\| x_1 \| \leq \| L^{-1}P_1 \| [\sup_{x \in X} \| N(x) \| + \| P_1 f \|] \equiv M$

and thus by hypothesis, there exists R_0 such that $P_0 N(a \cdot x_0 + x_1) \neq 0$ for $\| x_1 \| \leq M$ and $|a| \geq R_0$ so that on the bounded set $\{(x_1,a): \| x_1 \| \leq M, R_0 \leq |a| \leq R_0 + M\}$ there is some constant $\alpha > 0$ such that $\| P_0 N(a \cdot x_0 + x_1) \| > \alpha$. Thus picking $c = \alpha$, if $\| P_0 f \| < c$ and $k = M + R_0$ we have $H(x_1,a,t) \neq 0$. This gives us that $H(x_1,a,0)$ is homotopic to $H(x_1,a,1)$ on S_k. But $H(x_1,a,1) = F(x_1,a)$ and

(5) $H(x_1,a,0) = (x_1, P_0 N(a \cdot x_0) - P_0 f)$

so that

$\deg_{LS}(F,(0,0),D_k) = \deg (P_0 N(a \cdot x_0) - P_0 f, 0, D_k^n)$

$= \deg (\phi, 0, D_k^n) \neq 0$ by hypothesis (H.4).

It is easily seen from (4) and the subsequent inequalities that c and k depend on $P_1 f$.

Lemma 2. If $P_0 f \neq 0$, there is a k_1 depending on $P_0 f$

such that $\deg_{LS}(F,(0,0),D_{k_1}) = 0$.

Proof. Let $k_1 = M + \rho$ where M is given by equation (4) and ρ is given by hypothesis (H.1) with $\varepsilon = \|P_o f\|$.

Thus on S_{k_1}

$$G(x_1,a,t) = (x_1 + tL^{-1}P_1[N(a\cdot x_o + x_1) - f],$$

$$tP_o(a\cdot x_o + x_1) - P_o f)$$

is a non-vanishing homotopy between $F(x_1,a)$ and $G(x_1,a,0) = (x_1, -P_o f)$. But clearly

$$\deg_{LS}(G,(0,0),D_{k_1}) = 0$$

since G is not surjective. Thus $\deg_{LS}(F,(0,0),D_{k_1}) = 0$.

Finally we have

Proof of Theorem 1. Given f, suppose $\|P_o f\| < c$, where c is given in Lemma 1. Then there exists k such that $\deg_{LS}(F,(0,0),D_k) \neq 0$. But by Lemma 2, there is a k_1 such that $\deg_{LS}(F,(0,0),D_{k_1}) = 0$. Therefore there must be a zero of F between S_k and S_{k_1}. Thus we conclude that for $\|P_o f\| < c$, F must have at least two zeros.

Remark. Note that if $P_o f = 0$, the proof of Lemma 2 breaks down, and in fact Prof. Fučík has pointed out to me that the boundary-value problem with $f = 0$

$$-u'' - u + u(1 + u^2)^{-1} = 0$$

$$u(0) = u(\pi) = 0$$

satisfying (H.1) and (H.2), is uniquely solvable.

I would like to express my thanks to Prof. Fučík for the current formulation of hypothesis (H.1).

R e f e r e n c e s

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