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COVERING OF A SPACE BY NOWHERE DENSE SETS

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Abstract: The estimate of the cardinality of a family of nowhere dense sets which can cover a topological space without isolated points is given by means of cofinal subsets of ordinal-valued functions from cardinals. This improves some of known results.

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Definition. Let X be a dense-in itself topological space, ND(X) the set of all nowhere dense subsets of X. Define $n(X) = \min \{ |\mathcal{D}| : \mathcal{D} \subset ND(X) \ \& \ \cup \mathcal{D} = X \}$, and call this cardinal invariant the Novák number of a space X.

Let us recall several known facts about the Novák number:

(a) (Štěpánek-Vopěnka [ŠV]): If X is a nowhere separable metric space, then $n(X) = \omega_1$.

(b) (Broughan [B]): If X is dense-in-itself metric space, then $n(X) \le c$.

(c) (Štěpánek-Vopěnka [ŠV]): Let X be a uniformizable space, let α, β be cardinals such that $\omega \le \alpha < \alpha^+ \le \beta$ and suppose that

1. X admits a uniformity whose base $\mathcal{U}$ is linearly

ordered system of neighborhoods of diagonal with $|\mathcal{U}| = \alpha$, and

2. each non-void open subset of X contains at least β pairwise disjoint non-void open subsets.

Then $n(X) \leq \alpha^+$.

(d) (Kulpa-Szymański [KS]): Let $\alpha < \beta$ be cardinal numbers, β infinite and regular, and let X be a topological space satisfying the following:

1. X has a π-base $\mathcal{P}$ expressible as a union of α disjoint families, and

2. each non-void open subset of X contains at least β pairwise disjoint non-void open subsets.

Then $n(X) \leq \beta$.

The purpose of the present note is to prove the theorem, which is the common generalization of all results above, which gives a sharper bound for n(X) in some special cases and which can estimate n(X) for many spaces X where the above theorems are inapplicable.

Recall the following well-known notion: If $(P,<)$ is a partially ordered set and if $K \subset P$, then K is called cofinal in P iff for each $p \in P$ there is a $k \in K$ with $p < k$. The number cf(P) is then defined to be $\inf\{|K| : K \text{ is cofinal in } P\}$.

Consider, as usually, a cardinal number as an initial ordinal, ordered by $\in$. The set of all functions $f: \alpha \longrightarrow \beta$ (α, β cardinals) is denoted by ${}^{\alpha}\beta$ and ordered by $f < g$ iff $f(\xi) \in g(\xi)$ for all $\xi \in \alpha$. The number $cf({}^{\alpha}\beta)$ is then taken with respect to the order just described.

<u>Definition</u>. If X is a set, $\mathcal{A} \subset \mathcal{P}(X)$ and $x \in X$, then

$pc(\mathcal{A},x)$ is, by definition, $|\{A \in \mathcal{A} : x \in A\}|$ and $pc(\mathcal{A}) = \sup\{pc(\mathcal{A},x): x \in X\}$.

Now we are prepared to state the main result:

<u>Theorem</u>. Let X be a topological space and let α, β be cardinal numbers, β infinite, such that the following are true:

(i) X has a π-base $\mathcal{B}$ expressible as a union $\bigcup\{\mathcal{B}_\xi : \xi \in \alpha\}$, where $pc(\mathcal{B}_\xi) < cf(\beta)$ for each $\xi \in \alpha$,

(ii) to each $B \in \mathcal{B}$ one can assign a family $\{B(\eta) : \eta \in \beta\}$ of non-void open subsets of B with $pc\{B(\eta) : \eta \in \beta\} < cf(\beta)$.

Then $n(X) \leq cf({}^\alpha\beta)$.

<u>Remark</u>. It is clear that (d) is a special case of our theorem: it suffices to take $\mathcal{B} = \mathcal{P}$ and notice that the choice $\alpha < \beta$ with β regular implies $cf({}^\alpha\beta) = \beta$. (a) and (c) can be easily deduced from (d); the implication (d) $\longrightarrow$ (a) has already been established in [KS]. The proof of (b) goes as follows: Each metrizable space has a σ-discrete base, each non-void open subset in a dense-in-itself Hausdorff space contains infinitely many disjoint open non-void subsets, so the choice $\alpha = \beta = \omega$ is all right and $cf({}^\omega\omega)$ cannot be greater than c.

<u>Proof of the Theorem</u>. Let $\alpha, \beta, \mathcal{B}, \mathcal{B}_\xi$ $(\xi \in \alpha)$, $B(\eta)$ $(B \in \mathcal{B}, \eta \in \beta)$ be given as assumed in the theorem. For $\xi \in \alpha$ and $\eta \in \beta$ define $X_{\xi,\eta} = X - \bigcup\{B(\iota) : \eta \in \iota \in \beta,\ B \in \mathcal{B}_\xi\}$. The proof is a series of five easy observations, starting with an obvious

Observation 1: Each $X_{\xi,\eta}$ is closed.

For $f \in {}^\alpha\beta$ let $X_f = \bigcap\{X_{\xi,f(\xi)} : \xi \in \alpha\}$. As an intersection of closed sets, each X_f is closed.

Observation 2. For each $f \in {}^{\alpha}\beta$, X_f is nowhere dense. Let $\emptyset \neq U \subset X$ open be given. $\mathcal{B}$ is a σ-base, so one can find some $\xi \in \alpha$ and a $B \in \mathcal{B}_\xi$ with $\emptyset \neq B \subset U$. For $(\iota\, f(\xi)$, $\iota \in \beta$, by definition of $B(\iota)$, $\emptyset \neq B(\iota) \subset B \subset U$ and, by definition of $X_{\xi,f(\xi)}$, $B(\iota) \cap X_f \subset B(\iota) \cap X_{\xi,f(\xi)} = \emptyset$. Since U was chosen arbitrarily, X_f is nowhere dense.

Observation 3. Let $f,g \in {}^{\alpha}\beta$, $f < g$. Then $X_f \subset X_g$. (An obvious consequence of the definition $X_{\xi,\eta}$.)

Observation 4. For each $x \in X$ there is an $f \in {}^{\alpha}\beta$ with $x \in X_f$. Fix $x \in X$. For $\xi \in \alpha$ define $f(\xi) = \sup \{\eta \in \beta :$ there is a $B \in \mathcal{B}_\xi$ with $x \in B(\eta)\}$. Notice that the assumptions (i) and (ii) imply that the set of ordinals the sup is taken from is of cardinality less than $cf(\beta)$, thus $f \in {}^{\alpha}\beta$ is well-defined, because $f(\xi) \in \beta$. Clearly $x \in X_f$.

Combining the last two observations, we obtain immediately the final

Observation 5: If $K \subset {}^{\alpha}\beta$ is cofinal in ${}^{\alpha}\beta$, then $\bigcup \{X_f : f \in K\} = X$, which completes the proof.

<u>Corollary of the proof</u>: Let X, α, β satisfy the assumptions of the Theorem and suppose that ${}^{\alpha}\beta$ admits a well-ordered sequence (by $<$) of functions, which is cofinal and of cardinality $cf({}^{\alpha}\beta)$. Then X can be covered by a monotonically increasing sequence (of cardinality $cf({}^{\alpha}\beta)$) of nowhere dense sets.

(Use the Observation 3.)

<u>Examples</u>. A. A nowhere separable Souslin line L may

serve as an example of a space where (d) fails if one tries to estimate its Novák number. Recall that a Souslin line L is a connected LOTS with $c(L) = \omega$, $d(L) = \omega_1$. Since $\pi(X) \geq d(X)$ for any topological space, no π-basis for L is expressible as a union of less than ω_1 disjoint subfamilies, necessarily $\alpha \geq \omega_1$. On the other hand, no open subset of L admits more than countably many disjoint open subsets, thus $\beta \leq \omega$. Hence the assumptions of (d) can never be satisfied in this case.

It is widely known that a direct computation gives $n(L) \leq \omega_1$. Let us give another proof of this fact using our Theorem. Notice that L admits a π-basis $\mathcal{B}$ with $|\mathcal{B}| = \omega_1$ and $pc(\mathcal{B}) = \omega$. Set $\alpha = 1$, $\mathcal{B} = \mathcal{B}_0$ $(= \bigcup \{\mathcal{B}_\xi : \xi < 1\})$, and assign to each $B \in \mathcal{B}$ the family $\{B(\eta) : \eta < \omega_1\} = \{B' \in \mathcal{B} : B' \subset B\}$. The Theorem applies: $n(L) \leq cf({}^1\omega_1) = \omega_1$.

B. The inequality $pc(\mathcal{B}_\xi) < cf(\beta)$ cannot be replaced by $pc(\mathcal{B}_\xi) \leq cf(\beta)$ in (i) of the Theorem. As usual, denote by N^* the space $\beta N - N$, where N is a countable discrete set. Clearly $n(N^*) > \omega_1$ without any set-theoretical assumption.

But assume $c = \omega_{\omega_1}$, which is consistent with ZFC. Under this assumption N^* has a π-basis $\mathcal{B}$ such that $|\mathcal{B}| = c$ and $pc(\mathcal{B}) \leq \omega_1$, so let $\alpha = 1$, $\mathcal{B} = \mathcal{B}_0$. For $B \in \mathcal{B}$ let $\{B(\eta) : \eta < c\}$ be an arbitrary family of pairwise disjoint nonempty clopen subsets of B, thus $pc\{B(\eta) : \eta < c\} = 1$ for every $B \in \mathcal{B}$.

Applying the Theorem despite the fact that (i) is not satisfied, one has (remember that $c = \omega_{\omega_1}$) $n(N^*) \leq cf({}^1c) = cf(c) = \omega_1$, an obviously false result.

Remark. The referee has raised a question, whether there exists a space X such that $n(X) < cf({}^{\alpha}\beta)$ for every pair of cardinals α, β suitable for using the Theorem. Though the present author believes that such a space exists at least in some model of set theory, he regrets that he is not able to exhibit it.

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