COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

18,4 (1977)

NOTE TO PERIODIC SOLVABILITY OF THE BOUNDARY VALUE PROBLEM FOR NONLINEAR HEAT EQUATION

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Abstract: There is proved the existence of an ω-periodic solution of the boundary value problem for nonlinear heat equation. The proof is based on the Kazdan-Warner method (introduced for the solvability of boundary value problems for nonlinear partial differential equations of elliptic type) and on the theorem of Kolesov (where the existence of an ω-periodic solution of quasilinear parabolic equation follows from the existence of ω-periodic sub- and super-solutions).

Key words: Periodic solutions, nonlinear heat equation.

AMS: 35K05, 35K55 Ref. Ž.: 7.956

Let $\omega > 0$. Suppose that $f(t,x)$ is ω-periodic function in t. Let $\psi : R^1 \longrightarrow R^1$ be a given real valued function defined on the real line R^1. This note is devoted to the study of the existence of a solution of the problem

$$(1) \quad \begin{cases} u_t(t,x) - u_{xx}(t,x) - u(t,x) + \psi(u(t,x)) = f(t,x), \\ \qquad\qquad (t,x) \in Q = R^1 \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, \ t \in R^1 \\ u(t+\omega,x) = u(t,x), \ (t,x) \in Q. \end{cases}$$

In contrast to the previous results obtained for (1) by various authors (for an extensive bibliography see the prepared book of O. Vejvoda and Comp. [5]) our result will not be restricted to small nonlinearities although ψ will have to satisfy the monotonicity condition and certain one-side growth condition. The obtained result is in the spirit of a recent work by Kazdan-Warner [2] on boundary value problems for elliptic partial differential equations and may be generalized for higher dimensional analogue of the problem (1). The result is very close to Theorem V.1 from Brézis-Nirenberg [1], where the generalized solutions are considered and where also different one-side growth condition is supposed.

In the sequel we shall suppose:

(2) $f(t,x)$ is ω-periodic in the variable t and satisfies on $\overline{Q}$ the Hölder condition with some exponent $\alpha \in (0,1]$;

(3) the function $\psi : R^1 \longrightarrow R^1$ satisfies on arbitrary compact subinterval of R^1 the Hölder condition;

(4) the function ψ is nondecreasing on R^1 and there exists $c \geq 0$ such that

$$\psi(\xi) \geq -c\ (1 + \xi^2)$$

for arbitrary $\xi \in R^1$;

(5) $\lim_{\xi \to -\infty} \psi(\xi) < \psi(0) < \lim_{\xi \to \infty} \psi(\xi)$.

The continuous function $u^*(t,x)$ on $\overline{Q}$ is said to be a solution of (1) if it is ω-periodic in t, satisfies the boundary conditions (1_2), has the derivatives u_t^*, u_{xx}^* on Q and verifies the equation (1_1).

The main goal of this note is the following theorem.

__Theorem.__ Suppose (2) - (5). Then the problem (1) has at least one solution if and only if

$$(6) \quad 2\omega \lim_{\xi \to -\infty} \psi(\xi) < \int_0^{\omega}\int_0^{\pi} f(t,x) \sin x \, dx \, dt < 2\omega \lim_{\xi \to \infty} \psi(\xi).$$

The proof of Theorem

(i) Let (1) have a solution $u^*(t,x)$. Then

$$\int_0^{\omega}\int_0^{\pi} f(t,x) \sin x \, dx \, dt = \int_0^{\omega}\int_0^{\pi} \psi(u^*(t,x)) \sin x \, dx \, dt$$

and from the assumption (5) it follows the necessity of (6). (Note that for the using of the integration by parts we apply the regularity result that u^* is Hölder-continuous - see e.g. [4, Chap. 5, Thm. 1.1].)

(ii) Suppose (6). Then there exists a constant $k \in R^1$ such that $2\omega \ \psi(k)$ is close to

$$a = \int_0^{\omega}\int_0^{\pi} f(t,x) \sin x \, dx \, dt.$$

From the absolute continuity of the Lebesgue integral it is possible to perturb the constant k onto smooth function $z(x)$ on $[0,\pi]$ with $z(0) = z(\pi) = 0$ and such that

$$a = \omega \int_0^{\pi} \psi(z(x)) \sin x \, dx.$$

(The reader is invited to sketch a picture and to make a precise proof of the above assertion.)

(iii) Put

$P: (t,x) \longmapsto f(t,x) - \psi(z(x)), \ (t,x) \in \bar{Q}.$

Then for arbitrary continuously differentiable function u satisfying $(1_2),(1_3)$ and

$$z(x) \leq u(t,x), \ (t,x) \in \bar{Q}$$

we have

(7) $f(t,x) - \psi(u(t,x)) \leqq P(t,x), \quad (t,x) \in \overline{Q}.$

Analogously, for arbitrary continuously differentiable function $u(t,x)$ satisfying $(1_2),(1_3)$ and

$$u(t,x) \leqq z(x), \quad (t,x) \in \overline{Q}$$

it is

$$P(t,x) \leqq f(t,x) - \psi(u(t,x)), \quad (t,x) \in \overline{Q}.$$

(iv) The problem

$$(8) \quad \begin{cases} v_t - v_{xx} - v = P \text{ on } Q \\ v(t,0) = v(t,\pi) = 0, \quad t \in R^1 \\ v(t+\omega,x) = v(t,x) \text{ on } Q \end{cases}$$

has at least one solution $v^*(t,x)$ for

$$\int_0^{\omega}\int_0^{\pi} P(t,x) \sin x \, dx \, dt = 0.$$

Choose $\gamma \in R^1$ such that

(9) $\gamma \sin x + v^*(t,x) \geqq z(x), \quad (t,x) \in \overline{Q}.$

(Note that if $v(t,x)$ has continuous partial derivatives of the first order on $\overline{Q}$ and satisfies $(8_2),(8_3)$ then

$$\frac{|v(t\ \ )|}{\sin x} = \frac{|v(t,x) - v(t,0)|}{x} \cdot \frac{x}{\sin x} \leqq \sup_{x \in (0,\frac{\pi}{2})} \frac{x}{\sin x} \cdot$$

$$\cdot \sup_{(t,x) \in Q} |v_x(t,x)|$$

from which it follows (9) on $R^1 \times [0,\frac{\pi}{2})$ and analogously on $R^1 \times [\frac{\pi}{2}, \pi]$.)

Put

$$\overline{u}: (t,x) \longmapsto \gamma \sin x + v^*(t,x), \quad (t,x) \in \overline{Q}.$$

Then obviously $\overline{u}(t,x)$ satisfies $(1_2),(1_3)$ and from (7),(9) we have

$$\bar{u}_t(t,x) - \bar{u}_{xx}(t,x) - \bar{u}(t,x) + \psi(\bar{u}(t,x)) \geq f(t,x), \quad (t,x) \in Q.$$

Analogously, we choose $\delta \in R^1$ such that

$$\underline{u}: (t,x) \longmapsto \delta \sin x + v^*(t,x) \leq z(x), \quad (t,x) \in \bar{Q}.$$

Then $\underline{u}(t,x)$ satisfies $(1_2),(1_3)$ and

$$\underline{u}_t(t,x) - \underline{u}_{xx}(t,x) - \underline{u}(t,x) + \psi(\underline{u}(t,x)) \leq f(t,x), \quad (t,x) \in Q.$$

Obviously

$$\underline{u}(t,x) \leq \bar{u}(t,x), \quad (tx) \in \bar{Q}.$$

(v) The result of Kolesov (see [3]) implies that there exists at least one solution $u^*(t,x)$ of (1) which, moreover, satisfies

$$\underline{u}(t,x) \leq u^*(t,x) \leq \bar{u}(t,x), \quad (t,x) \in \bar{Q}.$$

R e f e r e n c e s

[1] H. BRÉZIS - L. NIRENBERG: Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, Ann. Scuola Norm. Sup. Pisa (to appear).

[2] J.L. KAZDAN - F.W. WARNER: Remarks on quasilinear elliptic equations, Comm. Pure Appl. Math. 28 (1975), 567-597.

[3] Ju. S. KOLESOV: Periodic solutions of quasilinear parabolic equations of the second order (Russian), Trudy Moskov. Mat. Obšč. 21(1970), 103-134.

[4] O.A. LADYŽENSKAJA, V.A. SOLONNIKOV, N.N. URALCEVA: Linear and quasilinear equations of the parabolic type (Russian), Moscow, Nauka, 1967.

[5] O. VEJVODA and Comp.: Partial differential equations - periodic solutions (manuscript of the prepared book).

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(Oblatum 18.8. 1977)