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NONLINEAR EQUATIONS WITH LINEAR PART AT RESONANCE: VARIATIONAL APPROACH

Svatopluk FUČÍK, Praha

Abstract: Under some assumptions we give the variational proofs of the existence results for the equation Lu = Su, where L is linear selfadjoint Fredholm and noninvertible, S is a nonlinear bounded and potential operator in a Hilbert space.

Key words: Potential operators, maxmin-points, minmax-points, nonlinear equations with linear part at resonance, implicit function theorem.

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1. Introduction. Let H be a real separable Hilbert space with the inner product $\langle u,v \rangle$ and with the norm $\|u\| = \langle u,u \rangle^{1/2}$. Suppose that $B: H \longrightarrow H$ is a linear completely continuous selfadjoint operator and let Λ be a sequence of all eigenvalues of B calculated together with the multiplicity. Let $e_\lambda \in H$, $\|e_\lambda\| = 1$, be the normalized eigenvector of B corresponding to $\lambda \in \Lambda$, i.e.

$$\lambda e_\lambda = Be_\lambda , \ \lambda \in \Lambda .$$

Choose the eigenvalue $\lambda_0 \in \Lambda$ to be fixed. Let W be a null-space of the operator

$$L: u \longmapsto \lambda_0 u - Bu, \ u \in H$$

and denote

d = distance of λ_0 to $\{\lambda \in \Lambda; \lambda \neq \lambda_0\}$.

Let $S: H \to H$ be strongly continuous nonlinear operator (i.e. it maps weakly convergent sequences $u_n \rightharpoonup u$ onto strongly convergent sequences $Su_n \to Su$) and suppose that

(1) $$\sup_{u \in H} \| Su \| = \alpha < \infty .$$

Moreover, let the operator S be potential with the potential $\mathcal{G} : H \to \mathbb{R}^1$ (i.e. the Fréchet derivative of $\mathcal{G}$ is S). Define

$$\sigma : r \longmapsto \sup_{\substack{w \in W \\ \|w\| = r}} \mathcal{G}(w),$$

$$\varkappa : r \longmapsto \inf_{\substack{w \in W \\ \|w\| = r}} \mathcal{G}(w).$$

The following result was firstly proved for partial differential operators in [1]; for abstract setting see [6], [3]. In [3] it is considered also the case of the growth condition

$$\| Su \| \leq \alpha + \beta \| u \|^{\delta} , \quad u \in H,$$

where $\delta \in (0,1]$ and they are given the applications to the boundary value problems for nonlinear partial differential equations and the existence theorems obtained by this way extend the previously proved results.

Theorem 1. Under the above assumptions the equation

(2) $$Lu = Su$$

is solvable in H provided one from the following conditions is satisfied:

(3) $$\lim_{r \to \infty} \sigma(r) = - \infty ,$$

(4) $$\lim_{r \to \infty} \varkappa(r) = \infty .$$

Theorem 1 is of the variational type, however, its proof is topological. In this note we shall give (under some additional assumptions) the variational proof of Theorem 1. The additional assumptions are important for the method used below and the obtaining of the variational proof of Theorem 1 without these assumptions is an open problem up to now.

We shall show that the solutions of (2) are characterized as maxmin-points (or minmax-points) of certain functional. This fact can be useful for using the numerical methods for constructing the solution of (2). Unfortunately, we had no success to obtain that the solutions of (2) are characterized as the saddle points of certain functional.

Before stating the main results of the present note let us introduce the following notation.

Let Z and V be the closures of linear hulls of all eigenvectors of B corresponding to $\lambda \in \Lambda$ with $\lambda > \lambda_0$ and $\lambda < \lambda_0$, respectively. Then $H = W \oplus V \oplus Z$ (the direct sum) and denote by $P_V, P_W, P_{W \oplus V}$ the orthogonal projections from H onto V, W and $W \oplus V$, respectively. Obviously

$$\langle Lv, v \rangle \geq d \| v \|^2, \quad v \in V,$$

$$\langle Lz, z \rangle \leq -d \| z \|^2, \quad z \in Z.$$

Define the functional $\Phi : W \times V \times Z \longrightarrow \mathbb{R}^1$ by

$$\Phi : (w,v,z) \longmapsto \frac{1}{2} \langle Lv, v \rangle + \frac{1}{2} \langle Lz, z \rangle - \mathcal{G}(w + v + z).$$

We shall seek a solution of (2) which satisfies

(5) $$\Phi(w_0,v_0,z_0) = \max_{z \in Z} \min_{(w,v)\in W\times V} \Phi(w,v,z)$$

or

(6) $$\Phi(w_0,v_0,z_0) = \min_{v \in V} \max_{(w,z)\in W\times Z} \Phi(w,v,z).$$

The main results are the following two theorems.

Theorem 2. Let $S: H \longrightarrow H$ be Fréchet differentiable at arbitrary $u \in H$. Suppose (1),(3) and that the Fréchet derivative $S'(u)$ at $u \in H$ (considered as a linear bounded operator from H into H) satisfies

(7) $$\langle S'(u)h,h \rangle < 0$$

for arbitrary $h \in H$, $h \neq 0$.

Then the equation (2) has at least one solution

(8) $$u_0 = w_0 + v_0 + z_0 \in H, \quad (w_0,v_0,z_0) \in W\times V\times Z$$

such that (5) holds.

On the other hand, arbitrary point $(w_0,v_0,z_0) \in W\times V\times Z$ satisfying (5) defines by the rule (8) the solution of (2).

Theorem 3. Suppose (1),(4) and that the Fréchet derivative $S'(u)$ at $u \in H$ satisfies

(9) $$\langle S'(u)h,h \rangle > 0$$

for arbitrary $h \in H$, $h \neq 0$.

Then the equation (2) has at least one solution (8) such that (6) holds. Arbitrary point $(w_0,v_0,z_0) \in W\times V\times Z$ satisfying (6) defines by the rule (8) the solution of (2).

The proof of Theorem 2 will be given in Section 2. The proof of Theorem 3 will be omitted as it is quite analogous to that of Theorem 2. In Section 3 we shall present some remarks concerning the special case

(10) $$\lambda_0 = \max \Lambda .$$

2. <u>The proof of Theorem 2</u>

(i) As S is strongly continuous the functional $\mathcal{S}$ is also strongly continuous. The assumption (7) implies that

(11) $$\langle Su_1 - Su_2, u_1 - u_2 \rangle < 0$$

for arbitrary $u_1, u_2 \in H$, $u_1 \neq u_2$, and that the functional $-\mathcal{S}$ is convex, i.e.

$$-\mathcal{S}(tu_1 + (1-t)u_2) \leq -t\,\mathcal{S}(u_1) - (1-t)\,\mathcal{S}(u_2)$$

for $u_1, u_2 \in H$, $t \in [0,1]$.

(ii) Notice that if $f: H \longrightarrow \mathbb{R}^1$ is Fréchet differentiable and convex on H then

$$f(u_0) = \min_{u \in H} f(u)$$

if and only if

$$f'(u_0) = 0.$$

(iii) Let $z \in Z$ be fixed. Then $\Phi(\cdot,\cdot,z)$ is two times Fréchet differentiable weakly lower semicontinuous (for definition see e.g. [4],[7]) on $W \times V$. From

$$\Phi(w,v,z) \geq \frac{d}{2}\|v\|^2 + \frac{1}{2}\langle Lz,z\rangle - \mathcal{S}(w+v+z) + \mathcal{S}(w) -$$
$$- \mathcal{S}(w) \geq \frac{d}{2}\|v\|^2 - \sigma(\|w\|) + \langle S(w + \vartheta(v+z)), v + z\rangle -$$
$$- \frac{1}{2}\|L\|\cdot\|z\|^2 \geq \frac{d}{2}\|v\|^2 - \sigma(\|w\|) - \alpha\|v\| - \alpha\|z\| -$$
$$- \frac{1}{2}\|L\|\cdot\|z\|^2$$

it follows that $\Phi(\cdot,\cdot,z)$ is coercive:

$$\lim_{\|w\|+\|v\|\to\infty} \Phi(w,v,z) = \infty .$$

From this and from the main theorem on calculus of variations (i.e. the lower weakly semicontinuous and coercive functional attains over reflexive Banach space its infimum - see e.g. [4],[7]) we obtain the existence of at least one couple $w(z)\in W$, $v(z)\in V$ such that

(12) $$\Phi(w(z),v(z),z) = \min_{(w,v)\in W\times V} \Phi(w,v,z).$$

(iv) <u>Lemma</u>. $w(z),v(z)$ with the property (12) are determined uniquely.

<u>Proof</u>. Suppose that there exist $z\in Z$ and $w_1,w_2\in W$, $v_1,v_2\in V$ such that

$$\Phi(w_1,v_1,z) = \Phi(w_2,v_2,z) = \min_{(w,v)\in W\times V} \Phi(w,v,z).$$

Then the partial Fréchet derivatives $\Phi_1'(w_i,v_i,z)$, $\Phi_2'(w_i,v_i,z)$ $(i = 1,2)$ vanish, i.e.

$$\langle Lv_i,k\rangle = \langle S(z + v_i + w_i),h + k\rangle$$

for $i = 1,2$ and arbitrary $h\in W, k\in V$. Put $k = v_1 - v_2$, $h = w_1 - w_2$. Then

$$d\|v_1 - v_2\|^2 \leqq \langle Lv_1 - Lv_2,v_1 - v_2\rangle =$$
$$=\langle S(w_1 + v_1 + z) - S(w_2 + v_2 + z), (w_1 - w_2) + (v_1 - v_2)\rangle \leqq 0$$

which implies $v_1 = v_2 = v$. Thus

$$\langle S(w_1 + v + z) - S(w_2 + v + z),w_1 - w_2\rangle = 0$$

from which together with (11) we get $w_1 = w_2$ and the uniqueness of $w(z), v(z)$ defined in (iii) is proved.

(v) Lemma. The mappings $w: Z \longrightarrow W$, $v: Z \longrightarrow V$ defined in (iii) map bounded subsets of Z onto bounded subsets of H.

Proof. It is

$d \| v(z) \|^2 \leq \langle Lv(z), v(z) \rangle = \langle S(z + w(z) + v(z)), v(z) \rangle \leq$

$\leq \alpha \| v(z) \|$. Thus if $M \subset Z$ is a bounded set then $\{v(z); z \in M\}$ is a bounded subset of V. From

$$\frac{d}{2} \|v(z)\|^2 - \sigma(\|w(z)\|) + \frac{1}{2}\langle Lz, z\rangle - \alpha \|v(z)\| -$$

$$- \alpha \|z\| \leq \Phi(w(z), v(z), z) \leq \Phi(0,0,z) \leq -\frac{d}{2}\|z\|^2 + \alpha \|z\|$$

it follows that $\{\sigma(\|w(z)\|); z \in M\} \subset \mathbb{R}^1$ is bounded from below and with respect to the assumption (3) the set $\{w(z); z \in M\} \subset W$ is bounded.

(vi) As $\Phi(.,.,z)$ is a convex functional on $W \times V$ we have with respect to (ii) that

$$v = v(z), \quad w = w(z)$$

if and only if

$$\langle Lv, k \rangle = \langle S(w + v + z), h + k \rangle$$

for arbitrary $h \in W$, $k \in V$.

(vii) Lemma. The mappings $w: Z \longrightarrow W$, $v: Z \longrightarrow V$ transform the weakly convergent sequences in Z onto strongly convergent sequences in H.

Proof. Let $\{z_n\}_{n=1}^{\infty} \subset Z$, $z_n \rightharpoonup z_0$ in Z. Then $\{\|w(z_n)\|\}_{n=1}^{\infty}$, $\{\|v(z_n)\|\}_{n=1}^{\infty}$ are bounded sequences of real numbers (see Lemma (v)) and with respect to the reflexivity of V and finite-dimensionalness of W there exists a subsequence $\{z_{n_i}\}_{i=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ such that

$$w(z_{n_i}) \longrightarrow w_o \text{ in } W, \quad v(z_{n_i}) \rightharpoonup v_o \text{ in } V.$$

Letting $i \longrightarrow \infty$ in

$$\langle Lw(z_{n_i}),k \rangle = \langle S(z_{n_i} + w(z_{n_i}) + v(z_{n_i})), h + k \rangle$$

we obtain

$$\langle Lv_o,k \rangle = \langle S(w_o + v_o + z_o),h + k \rangle$$

for arbitrary $h \in W$, $k \in V$. Thus according to (vi) we have

$$v_o = v(z_o), \quad w_o = w(z_o).$$

From this it easily follows (by contrary) that

$$v(z_n) \rightharpoonup v(z_o), \quad w(z_n) \longrightarrow w(z_o).$$

The strong convergence $v(z_n) \longrightarrow v(z_o)$ follows from

$$v(z_n) = KP_V\, S(w(z_n) + v(z_n) + z_n)$$

(where K is the inverse of L considered as an operator from V onto V) and from the strong continuity of S.

(viii) Lemma. The mappings $w: Z \longrightarrow W$, $v: Z \longrightarrow V$ are Fréchet differentiable.

Proof. Define $F: W \times V \times Z \longrightarrow W \times V$ by

$$F: (w,v,z) \longmapsto (-P_W S(w + v + z),\; P_V Lz - P_V S(w + v + z)).$$

Obviously

$$F'_{(w,v)}(w,v,z):(\tilde{w},\tilde{v}) \longmapsto (-P_W S'(w + v + z)(\tilde{w} + \tilde{v}),$$
$$P_V L\tilde{v} - P_V S'(w + v + z)(\tilde{w} + \tilde{v})).$$

According to (vi) and Implicit Function Theorem it is sufficient to prove that

$$F'_{(w,v)}(w(z),v(z),z) : W \times V \longrightarrow W \times V$$

is an isomorphism. Put

$$A: (\tilde{w},\tilde{v}) \longmapsto (\lambda_0 \tilde{w} + P_W S'(w + v + z)(\tilde{w} + \tilde{v}),\ P_V B\tilde{v} + $$
$$+ P_V S'(w + v + z)(\tilde{w} + \tilde{v})).$$

Obviously $A: W \times V \longrightarrow W \times V$ is completely continuous and

$$F'_{(w,v)}(w,v,z) : (\tilde{w},\tilde{v}) \longmapsto \lambda_0 (\tilde{w},\tilde{v}) - A(\tilde{w},\tilde{v}).$$

According to the Fredholm theory for linear operators to prove that $F'_{(w,v)}(w,v,z)$ is an isomorphism it is sufficient to prove that the equation

(13) $$P_V L\tilde{v} = P_{W \oplus V} S'(w + v + z)(\tilde{w} + \tilde{v})$$

has only a trivial solution. Let $(\tilde{w},\tilde{v}) \in W \times V$ be a solution of (13), Then $d \| v \|^2 \leq \langle L\tilde{v},\tilde{v} \rangle = \langle S'(w + v + z)(\tilde{w} + \tilde{v}),$ $\tilde{w} + \tilde{v} \rangle \leq 0$

and thus $\tilde{v} = 0$. From

$$P_{W \oplus V} S'(w + v + z)\, \tilde{w} = 0$$

we have

$$0 = \langle S'(w + v + z)\, \tilde{w},\tilde{w} \rangle$$

and the assumption (7) implies $\tilde{w} = 0$. The proof of the Fréchet differentiability of $w(z)$ and $v(z)$ is completed.

(ix) Define $G: Z \longrightarrow \mathbb{R}^1$ by

$$G: z \longmapsto \Phi(w(z),v(z),z).$$

Then $-G$ is weakly lower semicontinuous and

$$G(z) = \Phi(w(z),v(z),z) \leq \Phi(0,0,z) = \frac{1}{2}\langle Lz,z \rangle - \varphi(z) \leq$$
$$\leq -\frac{d}{2}\| z \|^2 + \alpha \| z \|$$

implies that -G is coercive, i.e.

$$\lim_{\|z\|\to\infty} G(z) = -\infty \ .$$

Thus there exists at least one $z_o \in Z$ such that

$$G(z_o) = \max_{z\in Z} G(z)$$

and we have $G'(z_o) = 0$, i.e.

$$\langle G'(z_o), z \rangle = 0$$

for arbitrary $z \in Z$. As

$$G'(z_o) = \Phi_1'(w(z),v(z),z) \circ w'(z) +$$
$$+ \Phi_2'(w(z),v(z),z) \circ v'(z) + \Phi_3'(w(z),v(z),z)$$

(the Leibniz rule on differentiation of composition) we have (putting $u_o = w(z_o) + v(z_o) + z_o$)

(14) $$0 = \langle G'(z_o),z \rangle = \langle Lv(z_o), v'(z_o)z \rangle + \langle Lz_o, z \rangle -$$
$$- \langle Su_o, w'(z_o)z + v'(z_o)z \rangle - \langle Su_o, z \rangle \ .$$

As

(15) $$\langle Lv(z_o), k \rangle = \langle Su_o, h + k \rangle$$

for arbitrary $k \in V$ and $h \in W$ we obtain from (14):

(16) $$\langle Lz_o, z \rangle = \langle Su_o, z \rangle$$

for arbitrary $z \in Z$. The relations (15) and (16) imply

$$Lu_o = Su_o$$

and the theorem is proved.

3. <u>Remarks</u>

(i) If (10) holds then $Z = \{0\}$ and if the assumptions of Theorem 2 are satisfied then the solution of (2) is unique.

(ii) If (10) holds then the additional assumptions upon S are not necessary. One can immediately prove the following theorem.

Theorem 4. Suppose (1),(3),(10). Then the equation (2) has at least one solution

(17) $u_0 = w_0 + v_0 \in H,\ w_0 \in W,\ v_0 \in V$

such that

(18) $\Phi(w_0, v_0) = \min\limits_{(w,v) \in W \times V} \Phi(w,v)$

(where $\Phi : (w,v) \longmapsto \frac{1}{2}\langle Lv, v\rangle - \mathcal{G}(w + v)$). On the other hand, arbitrary solution of (18) defines by the rule (17) the solution of (2).

(iii) Analogous result as in Theorem 4 is proved in [2] under more complicated assumptions.

(iv) The procedure how to prove Theorem 2 (or Theorem 3) extends the method from [5] for obtaining the existence of saddle point of convex-concave functional.

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Matematicko-fyzikální fakulta

Universita Karlova

Sokolovská 83, 18600 Praha 8

Československo

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