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REMARK TO THE CHARACTERIZATION OF THE SPHERE IN E^4

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Abstract: We get an example of the global characterization of the sphere in E^4 using the existence of a parallel vector field in the normal bundle of a surface.

Key words: Surface, parallel normal vector field, sphere.

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In [1], p. 62, A. Švec has mentioned one possibility of characterizing the sphere among the surfaces in E^4. In this contribution we give an example of the use of this idea. To give it, we have chosen one theorem, mentioned in [1], concerning the Weingarten surfaces in E^3, and translate it to the analogous theorem valid for surfaces in E^4.

Let M be a surface in the 4-dimensional Euclidean space E^4. Let the system of open sets $\{U_\alpha\}$ cover this surface in such a way that in any domain U_α there is a field of orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$ such that $v_1, v_2 \in T(M)$, $v_3, v_4 \in N(M)$ where $T(M)$, $N(M)$ is the tangent and normal bundle of M, respectively. Then

(1) $$dM = \omega^1 v_1 + \omega^2 v_2,$$

$$dv_1 = \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4, \quad dv_2 = -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4,$$

$$dv_3 = -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4,\ dv_4 = -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3;$$

(2) $$d\omega^i = \omega^j \wedge \omega_j^i,\ d\omega_i^j = \omega_i^k \wedge \omega_k^j,$$
$$\omega_i^j + \omega_j^i = 0,\ \omega^3 = \omega^4 = 0 \quad (i,j,k = 1,2,3,4).$$

Using the well-known prolongation process, we get the existence of real functions a_i, b_i ($i = 1,2,3$), α_i, β_i ($i = 1,2,3,4$) in each U_α such that

(3) $$\omega_1^3 = a_1\omega^1 + a_2\omega^2,\ \omega_2^3 = a_2\omega^1 + a_3\omega^2,$$
$$\omega_1^4 = b_1\omega^1 + b_2\omega^2,\ \omega_2^4 = b_2\omega^1 + b_3\omega^2;$$

(4) $$da_1 - 2a_2\omega_1^2 - b_1\omega_3^4 = \alpha_1\omega^1 + \alpha_2\omega^2,$$
$$da_2 + (a_1 - a_3)\omega_1^2 - b_2\omega_3^4 = \alpha_2\omega^1 + \alpha_3\omega^2,$$
$$da_3 + 2a_2\omega_1^2 - b_3\omega_3^4 = \alpha_3\omega^1 + \alpha_4\omega^2,$$
$$db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 = \beta_1\omega^1 + \beta_2\omega^2,$$
$$db_2 + (b_1 - b_3)\omega_1^2 + a_2\omega_3^4 = \beta_2\omega^1 + \beta_3\omega^2,$$
$$db_3 + 2b_2\omega_1^2 + a_3\omega_3^4 = \beta_3\omega^1 + \beta_4\omega^2.$$

Let $n = xv_3 + yv_4$ be a non-trivial, parallel normal vector field on M (see [11]). We can choose the field of orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$ in such a way that v_3 and n are dependent. Thus we have $y = 0$ and hence $dx = 0$, $x\omega_3^4 = 0$ on M, so that $\omega_3^4 = 0$ and

$$k = (a_1 - a_3)b_2 - (b_1 - b_3)a_2 = 0$$

on M.

Denote as usual

(5) $$H = (a_1 + a_3)^2 + (b_1 + b_3)^2,\ k = a_1a_3 - a_2^2 + b_1b_3 - b_2^2$$

the mean and Gauss curvature of M respectively, and define the functions

(6) $$H^1 = a_1 + a_3, \quad H^2 = b_1 + b_3$$

(7) $$K^1 = a_1a_3 - a_2^2, \quad K^2 = b_1b_3 - b_2^2.$$

Consider another field of orthonormal frames $\{M; \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ such that v_3 and n are dependent. Then

$$v_1 = \varepsilon_1 \cos\varphi . \bar{v}_1 - \sin\varphi . \bar{v}_2, \quad v_3 = \varepsilon_2 \bar{v}_3,$$

$$v_2 = \varepsilon_1 \sin\varphi . \bar{v}_1 + \cos\varphi . \bar{v}_2, \quad v_4 = \bar{v}_4, \quad \varepsilon_1^2 = \varepsilon_2^2 = 1.$$

We have according to [1]

$$\bar{a}_1 = \varepsilon_2[a_1\cos^2\varphi + 2a_2\sin\varphi\cos\varphi + a_3\sin^2\varphi],$$

$$\bar{a}_2 = -\varepsilon_1\varepsilon_2[(a_1 - a_3)\sin\varphi\cos\varphi + a_2(\sin^2\varphi - \cos^2\varphi)],$$

$$\bar{a}_3 = \varepsilon_2[a_1\sin^2\varphi - 2a_2\sin\varphi\cos\varphi + a_3\cos^2\varphi],$$

$$\bar{b}_1 = b_1\cos^2\varphi + 2b_2\sin\varphi\cos\varphi + b_3\sin^2\varphi,$$

$$\bar{b}_2 = -\varepsilon_1[(b_1 - b_3)\sin\varphi\cos\varphi + b_2(\sin^2\varphi - \cos^2\varphi)],$$

$$\bar{b}_3 = b_1\sin^2\varphi - 2b_2\sin\varphi\cos\varphi + b_3\cos^2\varphi$$

and it is easy to see that

$$\bar{H}^1 = \varepsilon_2 H^1, \quad \bar{H}^2 = H^2,$$

$$\bar{K}^1 = K^1, \quad \bar{K}^2 = K^2.$$

Now we are going to prove this

Theorem. *Let M be a surface in E^4 and ∂M its boundary. Let M satisfy these conditions:*

(i) $K > 0$ on M;

(ii) *there is a non-zero parallel normal vector field in* N(M);

(iii) there are functions $F(x,y), G(x,y)$ such that

(8) $F_x^2 + xF_xF_y + yF_y^2 > 0,\ G_x^2 + xG_xG_y + yG_y^2 > 0$

for each (x,y) and

(9) $$F(H^1,K^1) = 0,\ G(H^2,K^2) = 0$$

on M;

(iv) ∂M consists of umbilical points.

Then M is a part of a 2-dimensional sphere in E^4.

Proof. We use the method of integral formula based on the Stokes theorem.

On M, consider the 1-form

$$\tau = [(a_1 - a_3)\alpha_2 + (b_1 - b_3)\beta_2 - a_2(\alpha_1 - \alpha_3) - b_2(\beta_1 - \beta_3)]\omega^1 + [(a_1 - a_3)\alpha_3 + (b_1 - b_3)\beta_3 - a_2(\alpha_2 - \alpha_4) - b_2(\beta_2 - \beta_4)]\omega^2.$$

Using (5) we get by exterior differentiation of

(10) $$d\tau = -[2J + (H - 4K)K - 2k^2]\omega^1 \wedge \omega^2$$

where

(11) $$J = \alpha_2(\alpha_2 - \alpha_4) + \alpha_3(\alpha_3 - \alpha_1) + \beta_2(\beta_2 - \beta_4) + \beta_3(\beta_3 - \beta_1).$$

Now, consider the equations (9). By differentiation of these relations we obtain

(12) $$P_1 dH^1 + Q_1 dK^1 = 0,\ P_2 dH^2 + Q_2 dK^2 = 0$$

where we denoted

$$P_1 = \partial F/\partial H^1,\ Q_1 = \partial F/\partial K^1,\ P_2 = \partial G/\partial H^2,\ Q_2 = \partial G/\partial K^2.$$

Using (4),(6) and (7) we have

$$dH^1 = (\alpha_1 + \alpha_3)\omega^1 + (\alpha_2 + \alpha_4)\omega^2,$$

$$dH^2 = (\beta_1 + \beta_3)\omega^1 + (\beta_2 + \beta_4)\omega^2,$$

$$dK^1 = (a_1\alpha_3 - 2a_2\alpha_2 + a_3\alpha_1)\omega^1 + (a_1\alpha_4 - 2a_2\alpha_3 + a_3\alpha_2)\omega^2,$$

$$dK^2 = (b_1\beta_3 - 2b_2\beta_2 + b_3\beta_1)\omega^1 + (b_1\beta_4 - 2b_2\beta_3 + b_3\beta_2)\omega^2$$

and hence the equations (12) yield

$$(13)\qquad \begin{aligned} &P_1(\alpha_1 + \alpha_3) + Q_1(a_1\alpha_3 - 2a_2\alpha_2 + a_3\alpha_1) = 0,\\ &P_1(\alpha_2 + \alpha_4) + Q_1(a_1\alpha_4 - 2a_2\alpha_3 + a_3\alpha_2) = 0,\\ &P_2(\beta_1 + \beta_3) + Q_2(b_1\beta_3 - 2b_2\beta_2 + b_3\beta_1) = 0,\\ &P_2(\beta_2 + \beta_4) + Q_2(b_1\beta_4 - 2b_2\beta_3 + b_3\beta_2) = 0. \end{aligned}$$

Let $m \in M$ be an arbitrary fixed point of M. Consider that the orthonormal frame of M in the point $m \in M$ is chosen in such a way that $a_2 = 0$. Then we can put $b_2 = 0$ at $m \in M$ and the equations (13) have at $m \in M$ the form

$$\begin{aligned} &(P_1 + a_3Q_1)\alpha_1 + (P_1 + a_1Q_1)\alpha_3 = 0,\\ &(P_1 + a_3Q_1)\alpha_2 + (P_1 + a_1Q_1)\alpha_4 = 0,\\ &(P_2 + b_3Q_2)\beta_1 + (P_2 + b_1Q_2)\beta_3 = 0,\\ &(P_2 + b_3Q_2)\beta_2 + (P_2 + b_1Q_2)\beta_4 = 0. \end{aligned}$$

Thus, there are functions φ_i, σ_i $(i = 1,2)$ such that at $m \in M$

$$\begin{aligned} &\alpha_1 = \varphi_1(P_1 + a_1Q_1), \quad \alpha_3 = -\varphi_1(P_1 + a_3Q_1),\\ &\alpha_2 = \sigma_1(P_1 + a_1Q_1), \quad \alpha_4 = -\sigma_1(P_1 + a_3Q_1),\\ &\beta_1 = \varphi_2(P_2 + b_1Q_2), \quad \beta_3 = -\varphi_2(P_2 + b_3Q_2),\\ &\beta_2 = \sigma_2(P_2 + b_1Q_2), \quad \beta_4 = -\sigma_2(P_2 + b_3Q_2) \end{aligned}$$

and hence from (11)

$$J = \alpha_2^2 + \alpha_3^2 + \beta_2^2 + \beta_3^2 + (\rho_1^2 + \sigma_1^2)(P_1^2 + H^1P_1Q_1 + $$
$$+ K^1Q_1^2) + (\rho_2^2 + \sigma_2^2)(P_2^2 + H^2P_2Q_2 + K^2Q_2^2)$$

at $m \in M$.

The assumption (ii) implies $k = 0$ on M as mentioned, i.e. relation (10) has the form

$$d\tau = -[2J + (H - 4K)K]\,\omega^1 \wedge \omega^2.$$

Further on, from the condition (iv) it follows that $\tau = 0$ on ∂M. Thus, the Stokes integral formula yields

$$(14) \qquad \int_M [2J + (H - 4K)K]\,\omega^1 \wedge \omega^2 = \int_{\partial M} \tau = 0.$$

As $J \geq 0$ at $m \in M$ because of (i),(iii) and m is arbitrary, we have from (14)

$$2J + (H - 4K)K = 0$$

on M and hence

$$H - 4K = (a_1 - a_3)^2 + (b_1 - b_3)^2 + 4a_2^2 + 4b_2^2 = 0.$$

Thus any point m M is umbilical; this completes our proof.

Remark that we proved in fact a more general assertion which is obtained from the theorem replacing the assumption (iii) by

(iii') <u>there are functions</u> $P_i, Q_i : M \to \mathbb{R}$ $(i = 1,2)$ <u>such that</u>

$$P_1^2 + H^1P_1Q_1 + K^1Q_1^2 > 0, \quad P_2^2 + H^2P_2Q_2 + K^2Q_2^2 > 0$$

<u>and</u>

$$P_1dH^1 + Q_1dK^1 = 0, \quad P_2dH^2 + Q_2dK^2 = 0$$

<u>on</u> M.

In the following, we introduce three corollaries implied immediately by the proved theorem.

Corollary 1. *Let* M *be a surface in* E^4 *satisfying the conditions* (i), (ii) *and* (iv) *of the theorem. Let further*

(iii) H^1 = const , H^2 = const. *on* M.

Then M *is a part of a* 2-*dimensional sphere in* E^4.

The assertion follows from the theorem by putting $F(H^1,K^1) \equiv H^1$ - const., $G(H^2,K^2) \equiv H^2$ - const..

Corollary 2. *Let* M *be a surface in* E^4 *possessing the properties* (ii), (iv) *of the theorem. Let*

(i) $K^1 > 0$, $K^2 > 0$ *on* M;

(iii) H^1 = const., K^2 = const. (*or* H^2 = const., K^1 = = const.) *on* M.

Then M *is a part of a* 2-*dimensional sphere in* E^4.

To prove this, it is sufficient to put $F(H^1,K^1) \equiv H^1$ - - const., $G(H^2,K^2) \equiv K^2$ - const. (or $F(H^1,K^1) \equiv K^1$ - const., $G(H^2,K^2) \equiv H^2$ - const.) in the theorem.

Corollary 3. *Let* M *be a surface in* E^4 *satisfying* (ii), (iv) *of the theorem. Let*

(i) K^1 = const. > 0, K^2 = const. > 0 *on* M.

Then M *is a part of a* 2-*dimensional sphere in* E^4.

Let $F(H^1,K^1) \equiv K^1$ - const., $G(H^2,K^2) \equiv K^2$ - const.; then the assertion follows immediately from the theorem.

R e f e r e n c e s

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