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ON THE SPACE AND DUAL SPACE OF FUNCTIONS REPRESEBTABLE BY DIFFERENCES OF SUBHARMONIC FUNCTIONS

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Abstract: The linear space of differences of subharmonic functions is given a Fréchet space topology. This space together with its dual space is studied. A decomposition theorem for functionals vanishing on the harmonic functions is given and the functionals which are carried by one point is determined. It follows that, in the subharmonic case, the stable polar set always is countable.

Key words: Subharmonic function, Fréchet space, positive functional, polar set of a function.

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1. Introduction. Let U be an open subset of $\mathbb{R}^n$, $n \geq 2$, and denote by SH(U) the subharmonic functions on U. In this paper we study the linear space δSH(U), of functions which can be written as a difference of subharmonic functions. This subject has been treated by Arsove [1] and Kiselman [3]. We shall also study its dual space δSH'(U) here.

The corresponding function spaces made up by differences of convex or plurisubharmonic functions have been studied by Kiselman [3] and Cegrell [2].

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2. $\delta SH(U)$ and $\delta SH'(U)$. $\delta SH(U)$ is a Fréchet space with topology given by the seminorms $\| \varphi \|_K = \inf \left(\int_K |\varphi|_1 + |\varphi|_2 \;;\; \varphi = \varphi_1 - \varphi_2,\ \varphi_1, \varphi_2 \in SH(U) \right)$, $K \subset\subset U$. For a proof of this, see Schaefer [4], p. 221.

An equivalent topology on $\delta SH(U)$ is given by the seminorms

$$|||\varphi|||_{U'} = \inf \left(\int_{U'} |\varphi_1| + |\varphi_2| ;\ \varphi = \varphi_1 - \varphi_2,\ \varphi_1, \varphi_2 \in SH(U') \right)$$

where U' is open and relatively compact in U.

That $\delta SH(U)$ is complete under this topology is a consequence of Theorem 2.1. Moreover, $|||\cdot|||_{U'}$ gives a weaker topology than $\|\cdot\|_K$ and since both turn $\delta SH(U)$ into a Fréchet space, they are equivalent.

Theorem 2.1. If $\varphi \in L^1_{loc}(U)$ and if $\varphi|_{U'} \in \delta SH(U')$ for every U' open and relatively compact in U then $\varphi \in \delta SH(U)$.

Proof. Arsove [1] Theorem 10.

Definition. A compact subset K of U is said to be a carrier for $\mu \in \delta SH'(U)$ if to every open U' containing K there is a constant c such that

$$|\mu(\varphi)| \leq c \, \|\varphi\|_{U'} \qquad \forall \varphi \in \delta SH(U).$$

Definition. A subset K of U is said to be a support for $\mu \in \delta SH'(U)$ if, for any open O with $K \subset\subset O$, μ vanishes on those functions in $\delta SH(U)$ which vanish on $U \cap O$.

Definition. E is a notation for the fundamental solution to the equation $\Delta f = \delta_0$ in $\mathbb{R}^n$ when δ_0 is the Dirac measure at zero

$$E(z) = \begin{cases} \frac{1}{2\pi} \log |z|, & n = 2 \\ c_n \frac{-1}{|z|^{n-2}} & n > 2. \end{cases}$$

Corollary 2.2. Every $\mu \in \sigma SH(U)$ has a compact support.

Proof. Let B be a carrier for μ and choose U' open with $B \subset\subset U' \subset\subset U$. Then there is a constant $c > 0$ so that $|\mu(\varphi)| \leq c \|\varphi\|_{U'}$, $\forall \varphi \in \sigma SH(U)$. Since an equivalent topology on $\sigma SH(U)$ is defined by the seminorms $|||\ \ |||$ there is a constant d and an open U'' relatively compact in U so that $\|\varphi\|_{U'} \leq d\ |||\varphi|||_{U''}$ $\forall \varphi \in \sigma SH(U)$. Hence $|\mu(\varphi)| \leq$ $\leq c.d\ |||\varphi|||_{U''}$ $\forall \varphi \in \sigma SH(U)$ so if $\varphi \in \sigma SH(U)$ with $\varphi|_{U''} = 0$ then $\mu(\varphi) = 0$, which means that U'' is a support for μ.

Corollary 2.3. $\sigma SH(\mathbb{R}^n)|_U$ is dense in $\sigma SH(U)$.

Proof. By the Hahn-Banach theorem it is enough to prove that if $\mu \in \sigma SH'(U)$ vanishes on $\sigma SH(\mathbb{R}^n)|_U$ then $\mu \equiv 0$. So let $\varphi \in \sigma SH(U)$ and $\mu \in \sigma SH'(U)$ vanishing on $\sigma SH(\mathbb{R}^n)|_U$ be given. Choose $\theta \in \mathcal{D}(U)$, $0 \leq \theta \leq 1$ with $\theta = 1$ near a compact support A, for μ. Then $\varphi - E * \theta \Delta \varphi$ is subharmonic on U and harmonic near A. So there is a $\psi \in \sigma\ CVX(\mathbb{R}^n)$ with $\psi = \varphi - E * \theta \Delta \varphi$ near A. (See Kiselman [3].) Hence

$$0 = \mu(\psi) = \mu(\varphi - E * \theta \Delta \varphi) = \mu(\varphi)$$

since $E * \theta \Delta \varphi \in SH(\mathbb{R}^n)$.

Theorem 2.4. Assume that $\mu \in \sigma SH'(U)$ and that A and B are compact supports for μ. Then $A \cap B$ is a support.

Proof. Given U_3 open and $\varphi \in \sigma SH(U)$ vanishing near $\overline{U}_3$

where $A \cap B \subset\subset U_3 \subset\subset U$. We have to prove that $\mu(\varphi) = 0$. Choose U_2 open so that $B \subset\subset U_2$; $\overline{U}_2 \cap A \cap \complement U_3 = \emptyset$. Choose U_1 open so that $B \subset\subset U_1 \subset\subset U_2$ and $\theta_1 \in \mathcal{D}(U_2)$ with $\theta_1 = 1$ near $\overline{U}_1$. Then $\varphi = E * \theta_1 \Delta \varphi + h$ near $\overline{U}_1$ where h is harmonic near $\overline{U}_1$ and where $E * \theta_1 \Delta \varphi$ is harmonic on an open set U_4 such that $A \subset\subset U_4 \subset\subset U$; $U_4 \cap \overline{U}_2 \cap \complement U_3 = \emptyset$.

Choose now $\theta_2 \in \mathcal{D}(U_2)$; $\theta_2 = 1$ near $\overline{U}_1$ so that $\theta_2 \cdot h \in \sigma CVX(\mathbb{R}^n)$ and $\theta_3 \in \mathcal{D}(U_4)$, $\theta_3 = 1$ near A. Then $\theta_3 \cdot E * \theta_1 \Delta \varphi \in \sigma CVX(\mathbb{R}^n)$ and we define $f, g \in \sigma CVX(\mathbb{R}^n)$ by

$$f = \theta_2 \cdot \theta_3 \cdot h; \quad g = \theta_3 \cdot E * \theta_1 \Delta \varphi .$$

On $U_1 \cap U_3$ we have $g + f = \theta_3 E * \theta_1 \Delta \varphi + \theta_3 \cdot h = \theta_3 \varphi = 0$ since φ vanishes on U_3 and since $\theta_3 = 0$ on $U_1 \cap \complement U_3$, $g + f = 0$ on U_1 which contains B. Hence

$$0 = \mu(f + g) = \mu(E * \theta_1 \Delta \varphi + \theta_2 \cdot h) = \mu(\varphi)$$

since $\theta_2 = 1$ near $\overline{U}_1$ and the proof is complete.

<u>Remark</u>. Theorem 2.4 and Corollary 2.2 prove that every $\mu \in \sigma SH'(U)$ has a smallest compact support.

<u>Definition</u>. Let K be a compact subset of U. Then $\hat{K}$ is defined by

$$\hat{K} = \{ z \in U;\ \varphi(z) \le \sup_{\xi \in K} \varphi(\xi) \quad \forall \varphi \in SH(U) \}$$

<u>Lemma 2.5</u>. <u>Let</u> K <u>be compact in</u> U. <u>Then</u> $\hat{K}$ <u>is compact in</u> U. <u>Given</u> $\varepsilon > 0$, <u>and</u> U_1 <u>an open neighbourhood of</u> $\hat{K}$. <u>Then there is a continuous and subharmonic function</u> φ <u>on</u> U <u>such that</u> $\varphi = 0$ <u>on</u> $\hat{K}$ <u>and</u> $\varphi \ge \varepsilon$ on $\complement U_1$.

<u>Proof</u>. Consider $\hat{K}_c = \{ z \in U;\ \varphi(z) \le \sup_K \varphi \quad \forall \varphi \in SH(U) \cap C(U) \}$. It is clear that $K \subset \hat{K} \subset \hat{K}_c$ and if $z_0 \in \partial U$

$$- E(z - z_0) > \sup_{\xi \in K} [-E(\xi - z_0)] \quad \text{for } |z - z_0| < d(K, \complement U)$$

so it follows that $\hat{K}_c$ is compact in U.

We claim that $\hat{K} = \hat{K}_c$. If $z_1 \notin \hat{K}$ then there is a $\varphi \in$ $\in SH(U)$ with $\varphi(z_1) > \sup_K \varphi$. Choose $\varphi_2 \in SH(U) \cap C(U)$ so that $\varphi_\varepsilon \searrow \varphi$, $\varepsilon \searrow 0$ on a compact set containing z_1 and K in its interior. Then there is an ε_0 so that $\sup_K \varphi_{\varepsilon_0} \leq$ $\leq \frac{1}{2}(\varphi(z_1) + \sup_K \varphi)$. But $\varphi_{\varepsilon_0}(z_1) \geq \varphi(z_1)$ so $z_1 \notin \hat{K}_c$.

To a given open set U_1 with $\hat{K} \subset\subset U_1 \subset\subset U$ it is easy to see that there are finitely many functions $\varphi_i \in SH(U) \cap C(U)$, $1 \leq i \leq m$ with $\sup_{1 \leq i \leq m} \varphi_i = 0$ on $\hat{K}$ and $\sup_{1 \leq i \leq m} \varphi_i \geq 1$ on $\complement U_1$.

<u>Proposition 2.6</u>. Let K be a carrier for $\mu \in \delta SH'(U)$. Then $\hat{K}$ is a support for μ.

<u>Proof</u>. Let K be a carrier for $\mu \in \delta SH'(U)$ with $K = \hat{K}$. Choose an open set U_1 so that $\hat{K} \subset\subset U_1 \subset\subset U$ and let $\varphi \in \delta SH(U)$ with $\varphi|_{U_1} = 0$ be given. We have to prove that $\mu(\varphi) = 0$. Since

$$\varphi = \varphi_1 - \varphi_2 = \varphi_1 - E * \chi_{U_1} \Delta \varphi_1 -$$
$$- (\varphi_2 - E * \chi_{U_1} \Delta \varphi_1)$$

we have a representation ψ_1 and ψ_2 of φ where ψ_1 and ψ_2 are continuous near K. Using Lemma 2.5 we can find an open set U_2, $K \subset\subset U_2 \subset\subset U_1$ and a continuous subharmonic function ψ so that

$$\inf_{\partial U_2} \psi > \sup_{U_2} - \psi_2$$
$$\sup_{U_3} \psi \leq \inf_{U_3} - \psi_2$$

where $K \subset\subset U_3 \subset\subset U_2$ so it follows that

$$g(z) = \begin{cases} \psi(z), & z \notin U_2 \\ \sup(\psi_1 - \psi_2), & z \in U_2 \end{cases}$$ is continuous on U and subharmonic near $\complement U_2$.

Now $\theta_1 = \psi_1 + g$, $\theta_2 = \psi_2 + g$ are subharmonic on U and $\theta_1 - \theta_2 = \varphi$ so since K is a carrier for μ we have

$$|\mu(\varphi)| \leq C_{U_3} \int_{U_3} |\theta_1| + |\theta_2| = 0$$

and the proof is complete.

<u>Corollary 2.7.</u> Let μ be a non-vanishing element in $\delta SH(U)$. If A and B are carriers for μ then $\hat{A} \cap \hat{B} \neq \emptyset$.

3. <u>Positive functionals on</u> $\delta SH(U)$

<u>Definition.</u> Denote by $\delta SH'_+(U)$ the set of elements in $\delta SH'(U)$ which only takes non-negative values on SH(U).

<u>Remark.</u> Any real-valued linear map which is defined on $\delta SH(U)$ and which is non-negative on SH(U) is continuous (see Proposition 1.1 in Cegrell [2]). In particular, we have the following

<u>Lemma 3.8.</u> <u>Let</u> K <u>be a compact subset of</u> U. <u>Then</u>

$$\delta SH(U) \ni \varphi \longmapsto \Delta\varphi\{K\} \quad (= \int_K \Delta\varphi)$$

<u>is an element in</u> $\delta SH'(U)$.

<u>Theorem 3.9.</u> $\mu \in \delta SH'(U)$. <u>Then the following conditions are equivalent.</u>

1) $\mu = \mu_1 - \mu_2$ <u>where</u> $\mu_1, \mu_2 \in \delta SH'_+(U)$;

2) <u>there is a compact subset</u>, K, <u>of</u> U <u>such that</u> $\mu(\varphi)$ <u>vanishes for all</u> $\varphi \in \delta SH(U)$ <u>which are harmonic near</u> K;

3) μ <u>vanishes on the harmonic functions;</u>

4) <u>there is a compact subset</u>, K, <u>of</u> U <u>and a constant</u> c <u>so that</u> $|\mu(\varphi)| \leq c\int_K \Delta\varphi \quad \forall \varphi \in SH(U)$.

<u>Proof</u>. 1) $\Longrightarrow$ 2). If $\mu \in \delta SH_+'(U)$, let K be a carrier for μ. Then $\hat{K}$ is a support for μ by Proposition 2.6. Given $\varphi \in \delta SH(U)$ which is harmonic near $\hat{K}$ we can construct $\varphi_1, \varphi_2 \in SH(U)$ so that $\varphi_1 = -\varphi_2 = \varphi$ near $\hat{K}$. Hence $0 \leq \mu(\varphi_1) = \mu(\varphi) = \mu(-\varphi_2) \leq 0$ so $\mu(\varphi) = 0$.

2) $\Longrightarrow$ 3) is trivial.

3) $\Longrightarrow$ 4). Denote with M(U) the Fréchet space of measures on U with topology defined by seminorms $\|t\|_K$ = total mass of f on K, $K \subset\subset U$.

Let $\mathcal{H}a(U)$ be a notation for the harmonic functions on U, which form a closed subspace of $\delta SH(U)$. Let now j be a notation for the mapping

$$\delta SH(U)/\mathcal{H}a(U) \ni \varphi \overset{j}{\longmapsto} \Delta\varphi \in M(U).$$

That j is continuous follows from Lemma 3.8. Furthermore, j is a bijection so j^{-1} is continuous since both $\delta SH(U)/\mathcal{H}a(U)$ and M(U) are Fréchet spaces.

Now since $\mu = 0$ on $\mathcal{H}a(U)$ we have

$$|\mu(\varphi)| \leq c \inf_{h \in \mathcal{H}a(U)} \|\varphi + h\|_K \qquad \forall \varphi \in \delta SH(U)$$

for a fixed constant c and compact set K. But j^{-1} is continuous so there is another constant d and another compact set L in U so that

$$\inf_{h \in \mathcal{H}a(U)} \|\varphi + h\|_K \leq d \inf_{\substack{\varphi = \varphi_1 - \varphi_2 \\ \varphi_1, \varphi_2 \in SH(U)}} \int_L \Delta\varphi_1 + \Delta\varphi_2 \quad \forall \varphi \in \delta SH(U).$$

In particular, $\quad |\mu(\varphi)| \leq c \cdot d \int_L \Delta\varphi \qquad \forall \varphi \in SH(U)$.

4) $\Longrightarrow$ 1). $\mu(\varphi) = c\int_K \Delta\varphi + \mu(\varphi) - c\int_K \Delta\varphi$ is the desired representation.

4. Functionals on SH(U) carried by one point. We shall now determine all functionals $\mu \in \delta SH'(U)$ which are carried by one point $z_0 \in U$. We can of course restrict ourselves to the case $z_0 = 0$. Then, if μ is carried by the origin, μ is also supported by the origin. Let B denote $\{z;\ |z| < 1\}$.

Lemma 4.10. *Let $\varphi \in SH(B)$ and assume that φ is bounded below in a neighbourhood of zero. Then $\mu(\varphi) = 0 \quad \forall \mu \in \delta SH'(B)$ which are carried by zero.*

Proof. Given $\varphi \in SH(B)$ bounded below near zero. Assume first that $\varphi \leq 0$ on $\{z;\ |z| \leq r\}$ where $0 < r < 1$. If we put

$$\psi_n = \begin{cases} \sup\left(\frac{1}{n^2} \log\left|\frac{z}{r}\right|, \varphi\right), & |z| \leq r \\ \frac{1}{n^2} \log\left|\frac{z}{r}\right|, & |z| > r \end{cases}$$

it follows that $\psi_n \in SH(B)$ and $\psi_n = \varphi$ near zero.

Put $\theta_N = \sum_{\nu=1}^{N} \psi_\nu$. Then $(M > N)$

$$\|\theta_M - \theta_N\|_{B'} = \int_{B'} \left|\sum_{N+1}^{M} \psi_n\right| \leq$$

$$\leq \left(\int_B \left|\log\left|\frac{z}{z}\right|\right|\right) \sum_{n=N+1}^{M} \frac{1}{n^2} \longrightarrow 0, \ \min(M,N) \longrightarrow +\infty$$

for every B' relatively compact in B.

So it follows that θ_N converges to a limit $\theta \in \delta SH(U)$. Now $\mu(\theta) = \lim_{N\to+\infty} \mu(\theta_N) = \lim_{N\to+\infty} N\mu(\varphi)$ which gives $\mu(\varphi) = 0$. If we apply this to $\varphi - \sup_{|z|<\frac{1}{2}} \varphi$ the lemma follows.

<u>Definition</u>. Let $s_z(U)$ be the functions in $\delta SH(U)$ which have a representation $\varphi = \varphi_1 - \varphi_2$ where $\varphi_1 + \varphi_2(z) > -\infty$ and let $\mathcal{S}_z(U)$ denote the closure of $s_z(U)$ in $\delta SH(U)$.

<u>Lemma 4.11</u>. <u>Let</u> $\mu \in \delta SH'(B)$ <u>be carried by zero. Then</u> $\mu(\varphi) = 0 \quad \forall \varphi \in \mathcal{S}_0$.

<u>Proof</u>. It is enough to prove that if $\varphi \in SH(B)$ with $\varphi(0) > -\infty$ then $\mu(\varphi) = 0$. Choose $\theta_n \in \mathcal{D}(B)$, $0 \leq \theta_n \leq 1$, $\theta_n \geq \theta_{n+1}$, $\theta_n = 1$ near zero, $\lim_{n\to\infty} \theta_n = 0$ outside zero. By Theorem 3.9 and Lemma 4.10 there is a constant c so that

$$|\mu(\varphi)| = |\mu(E * \theta_n \Delta\varphi)| \leq c \int_K \theta_n \Delta\varphi \longrightarrow 0, \ n \to \infty$$

since $\varphi(0) > -\infty$.

<u>Definition</u>. Denote by $T_z(\varphi)$ the functional

$$\delta SH(B) \ni \varphi \longmapsto \Delta\varphi\{z\} \quad (= \int_{\{z\}} \Delta\varphi).$$

<u>Lemma 4.12</u>. $T_0(\varphi) = 0 \Longleftrightarrow \varphi \in \mathcal{S}_0$.

<u>Proof</u>. $\Longleftarrow$) Clear by Lemma 4.11.

$\Longrightarrow$) Choose θ_n as in the proof of Lemma 4.11 and assume that $\varphi = \varphi_1 - \varphi_2 \in \delta SH(B)$ with $T_0(\varphi_1) = T_0(\varphi_2)$. Then

$$E * \theta_n \Delta\varphi_1 \nearrow E \cdot T_0(\varphi_1), \quad n \to +\infty$$

$$E * \theta_n \Delta\varphi_2 \nearrow E \cdot T_0(\varphi_2), \quad n \to +\infty .$$

Put

$\psi_1 = \varphi_1 - E \cdot T_0(\varphi_1)$, then $\psi_1, \psi_2 \in SH(B)$,

$\psi_2 = \varphi_2 - E \cdot T_0(\varphi_2)$

$\varphi = \psi_1 - \psi_2$ and

$\psi_1^n = \psi_1 - E * \theta_n \Delta\psi_1 \in s_0, \quad n \in \mathbb{N}$

$\psi_2^n = \psi_2 - E * \theta_n \Delta \psi_2 \in s_0, \quad n \in \mathbb{N}$.

For every compact subset K of U we have

$\| \psi_1 - \psi_2 - (\psi_1^n - \psi_2^n) \|_K \leq \| E * \theta_n \Delta \psi_1 \|_K +$

$+ \| E * \theta_n \Delta \psi_2 \|_K \leq \int_K (E \cdot T(\varphi_1) - E * \theta_n \Delta \varphi_1) dz +$

$+ \int_K (E \cdot T(\varphi_1) - E * \theta_n \Delta \varphi_2) dz \longrightarrow 0 \quad n \longrightarrow +\infty$, which means that $\varphi \in \mathcal{S}_0$.

<u>Theorem 4.13</u>. <u>Let</u> $\mu \in \delta SH'(B)$ <u>be carried by zero</u>. <u>Then</u>

$$\mu(\varphi) = \mu(E) \cdot T_0(\varphi) \qquad \forall \varphi \in \delta SH(B).$$

<u>Proof</u>. If $T_0(\varphi) = 0$ then $\varphi \in \mathcal{S}_0$ by Lemma 4.12 and we have $\mu(\varphi) = 0$ by Lemma 4.11. Thus $\mu = \alpha \cdot T_0$ for some constant α and since $T_0(E) = 1$ the theorem follows.

<u>Remark</u>. The notation of polar and stable polar set for plurisubharmonic functions were introduced in Kiselman [3]. The <u>polar set of a function</u> $f \in \delta SH(U)$ is

$$P(f) = \bigcap_{f_1, f_2} (\{z \in U;\ (f_1 + f_2)(z) = -\infty\};\ f = f_1 - f_2,\ f_1, f_2 \in SH(U))$$

and the <u>stable polar set of</u> f is

$P_*(f) = \bigcup_{\omega} \bigcap_{g \in \omega} P(g)$ (ω varies over the neighbourhood of f).

Now, $P_*(f) = \{z \in Z;\ T_z(f) \neq 0\}$ but $\{z \in U;\ T_z(f) \neq 0\}$ is a countable set so it follows that the stable polar set of any function $f \in \delta SH(U)$ is countable.

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