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TOLERANCE RELATIONS ON COMPLETE LATTICES

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Abstract: It is shown that each compatible tolerance relation T on a complete lattice L has a homotopy representation by means of two semicongruences induced by T on L.

Key words: Tolerance, homotopy representation.

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The purpose of this short paper is to show that each compatible tolerance relation on a complete lattice has the property of the homotopy type, i.e. a compatible tolerance relation T on a complete lattice L can be decomposed into two semicongruences on L and, on the other hand, expressed by means of these two semicongruences. The concept of homotopy suitable for the approach here was introduced by Petrescu in [4]. The other observations of this note are based on the characterization of compatible tolerance relations by means of τ-coverings and related mappings given by Chajda, Niederle and Zelinka in [1]. For other properties of tolerance relations on algebras the reader is referred to the recent paper [3] of Chajda and Zelinka and to the references therein.

Let $\mathcal{A} = \langle A,F \rangle$ be an algebra with the support A and the set F of fundamental operations. A tolerance relation T on the set A is a binary, reflexive and symmetric relation on A. T is compatible with $\mathcal{A}$, if for any n-ary operation $f \in F$, where n is a positive integer, and for arbitrary elements $a_1,\dots,a_n$, $b_1,\dots,b_n$ of satisfying $a_i T b_i$ for $i = 1,\dots,n$, we have $f(a_1,\dots,a_n) T f(b_1,\dots,b_n)$.

Let M be a non-empty set. The family $\mathcal{M} = \{M_\gamma, \gamma \in \Gamma\}$, where Γ is a subscript set, is called a covering of M by subsets, if and only if M_γ is for each $\gamma \in \Gamma$ a subset of M and $\bigcup_\gamma \{M_\gamma \mid \gamma \in \Gamma\} = M$. As usually, we suppose that $M_\gamma \neq M_\beta$ for $\gamma \neq \beta$, γ, $\beta \in \Gamma$. A covering $\mathcal{M} = \{M_\gamma, \gamma \in \Gamma\}$ of M is called a τ-covering of M, if and only if $\mathcal{M}$ satisfies the following two conditions

(i) if $\gamma_0 \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, then $M_{\gamma_0} \subseteq \bigcup_\gamma \{M_\gamma \mid \gamma \in \Gamma_0\} \Rightarrow \bigcap_\gamma \{M_\gamma \mid \gamma \in \Gamma_0\} \subseteq M_{\gamma_0}$;

(ii) if $N \subseteq M$ and N is not contained in any set from $\mathcal{M}$, then N contains a two-element subset of the same property.

The following lemma shows the connection between tolerance relations on M and τ-coverings of M [1, Thm. 1].

Lemma 1. Let M be a non-empty set. There exists then a one-to-one correspondence between tolerance relations on M and τ-coverings of M such that if T is a tolerance relation on M and $\mathcal{M}_T$ is the τ-covering of M corresponding to T, then any two elements of M are in the relation T if and only if there exists a set from $\mathcal{M}_T$ which contains both of them.

The second lemma [1, Thm. 3] illuminates the properties of compatible tolerances on algebras.

Lemma 2. Let $\mathcal{A} = \langle A,F\rangle$ be an algebra, T a tolerance on $\mathcal{A}$ and $\mathcal{M}_T$ a τ-covering of A corresponding to T. The tolerance T is compatible with $\mathcal{A}$, if and only if there exists an algebra $\mathcal{B} = \langle B,G\rangle$ with the following properties:

(i) there exists a one-to-one mapping $\varphi : F \longrightarrow G$ such that for any positive integer n and for each $f \in F$ the operation φf is n-ary if and only if f is n-ary;

(ii) there exists a one-to-one mapping $\chi : \mathcal{M}_T \longrightarrow B$ such that for each n-ary operation $f \in F$ and for any n + 1 elements $M_0, M_1, \dots, M_n$ from $\mathcal{M}_T$ the equality $\varphi f(\chi(M_1), \dots, \chi(M_n)) = \chi(M_0)$ implies that for any n elements $a_1, \dots, a_n$ of A such that $a_i \in M_i$, $i = 1, \dots, n$, the element $f(a_1, \dots, a_n) \in M_0$.

Let $\mathcal{A} = \langle A,F\rangle$ and $\mathcal{B} = \langle B,G\rangle$ be two algebras of the same type. Let n_0 be the maximum number n for which there exists an n-ary operation f on $\mathcal{A}$ and I the interval $[1,n_0]$. A family $\xi = [(\alpha_i)_{i\in I}; \beta]$ of mappings of A into B such that $\beta(f(a_1, \dots, a_n)) = f(\alpha_1(a_1), \dots, \alpha_n(a_n))$ for every $n \leq n_0$, $a_1, \dots, a_n \in A$, is called a homotopy of $\mathcal{A}$ into $\mathcal{B}$. The mappings α_i are called components of homotopy ξ and β the principal component of ξ. Moreover, it is shown that each α_i induces an equivalence relation on A [4, Lemma 0.1].

We shall show that the mapping χ relating to a compatible tolerance T on L is a principal component of a homotopy induced by T. The components α_1 and α_2 are generated by semicongruences on L which are induced by the τ-covering $\mathcal{M}_T$ of T. We shall construct α_1 which is given by an equivalence relation $E(\alpha_1)$ being compatible with respect to the

$\wedge$-operation on L, i.e. by a $\wedge$-semicongruence. The equivalence relation $E(\alpha_1)$ is constructed by determining the partition $\mathcal{E}$ inducing $E(\alpha_1)$. $\mathcal{E}$ is obtained by modifying the τ-covering $\mathcal{M}_T$ of the compatible tolerance T on L.

Theorem 1. Let T be a compatible tolerance on a complete lattice L, $\mathcal{M}_T$ the τ-covering corresponding to T and let $M_\delta \in \mathcal{M}_T$. Then the family of sets $\mathcal{M}_T^\wedge = \{M_\delta^\wedge, \delta \in \Gamma\}$, where $M_\delta^\wedge = M_\delta \setminus \bigcup_\gamma \{M_\gamma \mid M_\gamma \cap M_\delta \neq \emptyset$, the least element $e_{1\gamma}$ of M_γ is greater than the least element $e_{1\delta}$ of M_δ ($e_{1\delta} < e_{1\gamma}$), $M_\gamma \in \mathcal{M}_T\}$, forms a partition of L determining a $\wedge$-semicongruence on L.

Proof. According to [2, Thm. 1], each $M_\gamma \in \mathcal{M}_T$ is a conves sublattice of L, and as L is complete, there are in M_γ the least and greatest elements $e_{1\gamma}$ and $e_{g\gamma}$, respectively.

According to the definition of $M_\delta^\wedge$, each $M_\delta^\wedge$ contains at least $e_{1\delta}$, whence $M_\delta^\wedge \neq \emptyset$ for each $\delta \in \Gamma$. Moreover, the properties of T imply that when $a, b \in M_\delta^\wedge$ then also $a \wedge b \in M_\delta^\wedge$. Thus the theorem holds, if we can show that any element $x \in L$ belongs to at least one set $M_\delta^\wedge$ of $\mathcal{M}_T^\wedge$, and $M_\delta^\wedge \cap M_\gamma^\wedge = \emptyset$ for each pair $\delta, \gamma \in \Gamma$ when $\delta \neq \gamma$.

Let $a \in L$ and $\mathcal{M}_T(a)$ be the family of all subsets of $\mathcal{M}_T$ containing the element a. Let $M_\gamma, M_\varkappa \in \mathcal{M}_T(a)$ be such sets that $e_{1\gamma}$ and $e_{1\varkappa}$ are non-comparable. Let q be the least element of the set $M_\gamma \cap M_\varkappa$; such an element q exists and $q \in M_\gamma \cap M_\varkappa$, since L is complete and as an intersection of two convex sets $M_\gamma \cap M_\varkappa$ is a convex set of L, too. As $q \in M_\gamma$, $M_\varkappa$, $q \geq e_{1\gamma} \vee e_{1\varkappa}$ and as $e_{1\gamma} \vee e_{1\varkappa} \in M_\gamma \cap M_\varkappa$,

$q \leq e_{1\gamma} \vee e_{1\varkappa}$, whence $q = e_{1\gamma} \vee e_{1\varkappa}$. According to the compatibility of T, any two elements of the interval $[e_{1\gamma} \vee e_{1\varkappa}, e_{g\gamma} \vee e_{g\varkappa}]$ are in the relation T. Thus $[e_{1\gamma} \vee e_{1\varkappa}, e_{g\gamma} \vee e_{g\varkappa}] \subseteq M_\lambda \in \mathcal{M}_T(a)$ for some index $\lambda \in \Gamma$. If $e_{1\lambda} \leq e_{1\gamma}$, then $M_\gamma \notin \mathcal{M}_T$ according to the condition (i) for $\mathcal{M}_T$; the same holds for $M_\varkappa$, too. If $e_{1\gamma}$ and $e_{1\lambda}$ are non-comparable, then $(e_{1\gamma} \wedge e_{1\lambda}) T e_{g\gamma}$, as $e_{g\gamma} \in M_\gamma, M_\lambda$. Then $e_{1\gamma} > e_{1\gamma} \wedge e_{1\lambda}$, and so there were in $\mathcal{M}_T$ a set containing properly M_γ, which is a contradiction. Hence $e_{1\gamma}, e_{1\varkappa} \leq e_{1\lambda} \leq e_{1\gamma} \vee e_{1\varkappa}$ and so $e_{1\lambda} = e_{1\gamma} \vee e_{1\varkappa}$. Consequently, there is in $\mathcal{M}_T(a)$ for any two sets $M_\gamma, M_\varkappa$ a third set M_λ such that $e_{1\lambda} = e_{1\gamma} \vee e_{1\varkappa}$. As L is complete, there is also an element $\bigvee_\varkappa \{ e_{1\varkappa} \mid \varkappa$ goes over all indices of the sets in $\mathcal{M}_T(a) \} = e_{1\wp}$, where $e_{1\wp}$ is the least element of a subset $M_\wp$ belonging to the τ-covering $\mathcal{M}_T$ and containing the element a. According to the definition of $\hat{M}_\wp$ and to the maximality of $e_{1\wp}$ with respect to a, $a \in \hat{M}_\wp$, and so any element of L belongs to at least one of the sets $\hat{M}_\gamma$, $\gamma \in \Gamma$.

If $\hat{M}_\gamma \cap \hat{M}_\delta \neq \emptyset$, $\gamma \neq \delta$, then we can prove as above that $e_{1\delta} \vee e_{1\gamma} \in \hat{M}_\delta \cap \hat{M}_\gamma$. But this is the least element of a subset $M_\lambda \in \mathcal{M}_T$, $e_{1\lambda} > e_{1\delta}, e_{1\gamma}$, and thus, according to the definitions of $\hat{M}_\delta$ and $\hat{M}_\gamma$, $e_{1\lambda} \in \hat{M}_\delta, \hat{M}_\gamma$. This is a contradiction, whence $\hat{M}_\gamma \cap \hat{M}_\delta = \emptyset$ for any pair $\delta, \gamma \in \Gamma$, $\delta \neq \gamma$. This completes the proof.

Let T be a compatible tolerance on a complete lattice L and χ a mapping, $\chi : \mathcal{M}_T \rightarrow B$, induced by T and defined in Lemma 2. As for any $\gamma \in \Gamma$ there exists a unique subset

$M_\gamma^\wedge \in \mathcal{M}_T^\wedge$ of L, we can define a mapping $\alpha_1 : L \longrightarrow B$ as follows: for any $a \in L$, $\alpha_1(a) = \chi(M_\gamma)$ if and only if $a \in M_\gamma^\wedge$ in $\mathcal{M}_T^\wedge$.

By using the dual proof of Theorem 1, we can show the existence of a partition $\mathcal{M}_T^\vee$ of L determining a $\vee$-semicongruence on L. As above, we define the mapping $\alpha_2 : L \longrightarrow B$ induced by $\mathcal{M}_T^\vee$ and χ: for any $a \in L$, $\alpha_2(a) = \chi(M_\gamma)$ if and only if $a \in M_\gamma^\vee$ in $\mathcal{M}_T^\vee$. Now we are able to state our main theorem

<u>Theorem 2</u>. Let L be a complete lattice, T a compatible tolerance on L, $\mathcal{M}_T$ the corresponding τ-covering of L and χ the mapping, $\chi : L \longrightarrow B$, induced by T, where B is the carrier set of the algebra $\mathcal{B} = \langle B, G \rangle$ defined in Lemma 2. Then the triple $\xi = [\alpha_1, \alpha_2; \chi]$ determines a homotopy of L into $\mathcal{B} = \langle B, G \rangle$.

<u>Proof</u>. As χ is defined only on the family $\mathcal{M}_T$, we have to define χ on L such that it gives the desired homotopy property. For the two operations of L we define:

$\chi(f(a_1, a_2)) = \chi(M_0)$ which is obtained from $\varphi f(\chi(M_1), \chi(M_2))$, where $a_1 \in M_1^\wedge$ and $a_2 \in M_2^\vee$ (see Lemma 2 (ii)). As $a = a \vee a = a \wedge a$ in L, we obtain $\chi(a) = \chi(f(a,a))$ which is already defined. By using this definition for $\chi : L \longrightarrow B$ it obviously holds that $\chi(f(a_1, a_2)) = \varphi f(\alpha_1(a_1), \alpha_2(a_2))$ for any $a_1, a_2 \in L$, where φf can be substituted by f as L and $\mathcal{B}$ are of the same type. This completes the proof.

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