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K-ESSENTIAL SUBGROUPS OF ABELIAN GROUPS III

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Abstract: The purpose of this paper is to continue the investigation of K-essential subgroups of abelian groups begun in [1] and [2]. Essentially, the paper was motivated by some results of F.V. Krivonos [6]. Here, we give the necessary and sufficient conditions for a subgroup K of the group G to be the unique N-N∩K-high subgroup of G (see Theorem 1.1), or the unique N-high subgroup of G (Theorem 1.2). Further, there are investigated relations between N-K-high subgroups of G when the subgroups N and K vary.

Key words: K-essential and essential subgroups; N-high and N-K-high subgroups.

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0. Introduction. This paper develops the theory of K-essential subgroups as it was introduced in [1] and [2]. All groups considered in this paper are abelian. Concerning the terminology and notation we refer to [3],[4] and [1], [2].

If $K \subset N$ are subgroups of a group G then a subgroup A, which is maximal with respect to the property $A \cap N = K$ is said to be N-K-high in G (following Krivonos [6]).

For convenience, we are going to introduce the following definitions from [1] and [2].

Definition 0.1. Let K be a subgroup of a group G. A subgroup N of G is said to be K-essential in G (and G is said to be K-essential extension of N) if for each $g \in G \setminus K$ there exists an integer $n > 0$ with $ng \in N \setminus K$.

Definition 0.2. Let K be a subgroup of a group G. The set of all $g \in G$ such that there is a square-free integer $n > 0$ with $ng \in K$ we call K-socle of G and denote by G^K. Hence G^0 is the socle of G (see [2]).

Definition 0.3. Let N be a subgroup of a group G. We denote by $\mathcal{K}(N)$ the least subgroup from the set of all subgroups L such that N is L-essential in G (see [2]).

1. The uniqueness conditions. In this section there are given generalizations and supplements to the section 1 of [6], all results, presented there, immediately follow from Theorem 1.2. Further, Theorem 1.1 gives another condition equivalent to the assertions (i) - (vii), Proposition 1.3 [1].

Theorem 1.1. Let N and K be subgroups of a group G. Then K is the unique N-$N \cap K$-high subgroup of G iff either $K = G$ or G/N is torsion, $G^K \subset N+K$ and $(K/K \cap N)_p \neq 0$ implies $(N/K \cap N)_p = 0$.

Proof. Let K be the unique N-$N \cap K$-high subgroup of G. By 1.3 [1], N is K-essential in G. Suppose that $K \neq G$. Now, G/N is torsion by 3.2 [1], $G^K \subset N+K$ by 1.3 [2] and $(K/K \cap N)_p \neq 0$ implies $(N/K \cap N)_p = 0$ by 2.2 [1].

If $K = G$ then obviously K is the unique N-$N \cap K$-high subgroup of G. In the other case, $K/K \cap N$ is torsion, since G/N is so. If g is nonzero element of G then either $0 \neq ng \in N$ for some $n \in \mathbb{N}$ or $g \in G_t$. Let $\sigma(g) = pm$, where p is a prime and $m \in \mathbb{N}$. Now, $0 \neq mg \in G^K \subset N + K$ and the group $N + K$ is essential in G. Further, N is K-essential in $N + K$ by 2.2 [1] and $N + K$ is K-essential in G by 1.3 [2]. Hence N is K-essential in G by 1.5 [1] and K is the unique N-$N \cap K$-high subgroup of G by 1.3 [1].

Theorem 1.2. Let N and K be subgroups of a group G. The following are equivalent:

(i) K is the unique N-high subgroup of G;

(ii) N is K-essential in G and $K \cap N = 0$;

(iii) $K = \mathcal{K}(N)$ and $K \cap N = 0$;

(iv) Either N = 0 and K = G or there is a subset $\mathbb{R}$ of $\mathbb{P}$ such that $K = \bigoplus_{p \in \mathbb{R}} G_p$ and N is an essential subgroup of some K-high subgroup of G;

(v) Either N = 0 and K = G or G/N is torsion and there is a subset $\mathbb{R}$ of $\mathbb{P}$ such that $K = \bigoplus_{p \in \mathbb{R}} G_p$ and $N^0 = \bigoplus_{p \in \mathbb{P} \setminus \mathbb{R}} G[p]$.

Proof. The assertions (i) and (ii) are equivalent by 1.3 [1]. Further, (i) and (ii) imply $K = \mathcal{K}(N)$. Obviously, (iii) implies (ii).

(iii) $\Longrightarrow$ (iv),(v). By 4.3 [2], either K = G and N = 0 or G/N is torsion and there is a subset $\mathbb{R}$ of $\mathbb{P}$ such that $K = \bigoplus_{p \in \mathbb{R}} G_p$ and $\bigoplus_{p \in \mathbb{P} \setminus \mathbb{R}} G[p] \subset N$. According to the remark after Proposition 1.3 [1], N is essential in some K-high subgroup of G.

(iv) $\Longrightarrow$ (ii). If $K \neq G$ then for every $g \in G \setminus K$ there is $n \in \mathbb{N}$ with $ng \in N \setminus K$.

(v) $\Longrightarrow$ (iii). It follows from 4.3 [2].

Lemma 1.3. Let $N \subset N'$ be subgroups of a group G. Then the following are equivalent:

(i) A subgroup A of G is N-high in G iff A is N'-high in G;

(ii) There is an N-high subgroup K of G that is simultaneously N'-high in G;

(iii) N is essential in N'.

Proof: Easy.

Theorem 1.4. Let K be an N-high subgroup of a group G. Then N is the unique subgroup of G such that K is N-high in G iff $N = \bigoplus_{p \in \mathbb{R}} G_p$ for some subset $\mathbb{R}$ of the set $\{p \in \mathbb{P} ; G_p = G[p]\}$.

Proof. Let N be the unique subgroup of G such that K is N-high in G. Since K is M-high in G for every K-high subgroup M in G (see [5], p. 327), N is the unique K-high subgroup of G. By 1.2, either N = G or $N = \bigoplus_{p \in \mathbb{R}} G_p$, where $\mathbb{R}$ is some subset of $\mathbb{P}$. Further, N has no proper essential subgroup by 1.3 and hence N is elementary by 1.5 [2].

Conversely, suppose that $N = \bigoplus_{p \in \mathbb{R}} G_p$ for some subset $\mathbb{R}$ of the set $\{p \in \mathbb{P} ; G_p = G[p]\}$. Now, N is the unique K-high subgroup of G by 1.2. If K is M-high in G for some subgroup M of G then $M \subset N$. Since M is essential in N by 1.3, M = N by 1.5 [2].

2. *Comparing N-K-high and N-high subgroups*. This section, in general, investigates the problems arisen from Theorem 4, [6]. In particular, this theorem immediately follows from Proposition 2.5, presented in this section.

Lemma 2.1. Let $K \subset N$ and $K' \subset N'$ be subgroups of a group G. If there is a N-K-high subgroup of G that is contained in some N'-K'-high subgroup of G then $N' \cap K \subset N \cap K'$.

Proof: Easy.

Proposition 2.2. Let $K \subset N$ and $K' \subset N'$ be subgroups of a group G. Then every N-K-high subgroup of G is contained in some N'-K'-high subgroup of G iff for every $g \in N' \setminus K'$ there is $n \in \mathbb{N}$ with $ng \in N \cap N' \setminus K$.

Proof. Let every N-K-high subgroup of G be contained in some N'-K'-high subgroup of G. If $g \in N' \setminus K'$ then g is contained in no N-K-high subgroup of G. Hence $\langle K,g \rangle \cap N \supsetneq K$ and there is $n \in \mathbb{N}$ with $ng \in N \cap N' \setminus K$.

Conversely, let A be a N-K-high subgroup of G. If $g \in A \cap (N' \setminus K')$ then $ng \in N \cap N' \setminus K$ for some $n \in \mathbb{N}$. Hence $ng \in A \cap N = K$, a contradiction. Further, $A \cap N' \subset K'$ and consequently $(A + K') \cap N' = K'$. The group $A + K'$ is contained in some N'-K'-high subgroup of G.

Theorem 2.3. Let $K \subset N$ and $K' \subset N'$ be subgroups of a group G. Then the set of all N-K-high subgroups of G and the set of all N'-K'-high subgroups of G are identical iff

(i) $N \cap K' = N' \cap K$,

(ii) $N \cap N'$ is K-essential in N,

(iii) $N \cap N'$ is K'-essential in N'.

Proof. Suppose that the set of all N-K-high and N'-K'-high subgroups of G are identical. Hence $N' \cap K = N \cap K'$ by 2.1 and the assertions (ii) and (iii) follow from 2.2.

Conversely, assume that the conditions (i) - (iii) hold. Then $N \cap N' \setminus K = N \cap N' \setminus K'$. By 2.2, every N-K-high subgroup of G is contained in an N'-K'-high subgroup of G and conversely.

Proposition 2.4. Let K, K', N, N' be subgroups of a group G such that $K \subset K'$, $K \subset N$ and $K' \subset N'$. If every N-K-high subgroup of G is N'-K'-high in G then every N'-K'-high subgroup of G is N-K-high in G.

Proof. By 2.1 and 2.2, $N' \cap K = N \cap K'$ and $N \cap N'$ is K'-essential in N'. Let $g \in N \setminus K$. If A is an N-K-High subgroup of G then $\langle g, A \rangle \cap N' \supsetneq K'$. Hence there are $m \in N' \setminus K'$, $a \in A$ and $k \in \mathbb{N}$ such that $kg + a = m$. Now, there is $n \in \mathbb{N}$ such that $nkg + na = nm \in N \cap N' \setminus K$. Consequently, $na \in A \cap N = K \subset K'$ and $nkg \in N \cap N' \setminus K$. The rest follows from 2.3.

Let $K \subset N$ and H be subgroups of a group G. Then every N-K-high subgroup of G is contained in some H-high subgroup of G iff $H \cap K = 0$ and $H \cap N$ is essential in H.

Proposition 2.5. Let $K \subset N$ and H be subgroups of a group G. The following are equivalent:

(i) A subgroup A of G is H-high in G iff A is N-K-high in G;

(ii) Every H-high subgroup of G is N-K-high in G;

(iii) $H \cap K = 0$, $H \cap N$ is essential in H and K-essential in N;

(iv) Either $K = N$ and $H = 0$ or there is a subset $\mathbb{R}$ of $\mathbb{P}$ such that $K = \bigoplus_{p \in \mathbb{R}} N_p$ and $H \cap N$ is essential in some K-high subgroup of N and essential in H;

(v) The group K is contained in the intersection of all H-high subgroups of G and there is an essential subgroup H' of H such that N is a K-essential extension of H'.

Proof. The assertions (i) and (ii) are equivalent by 2.4, (i) and (iii) are equivalent by 2.3 and (iii) and (iv) by 1.2. The assertion (v) follows immediately from (i) and (iii). Conversely, (v) implies (iii).

Notice that the intersection of all H-high subgroups of G is equal the group $\bigoplus_{p \in \mathbb{R}} G_p$, where $\mathbb{R}$ is the set of all primes p with $H_p = 0$ (see [6], prop. 9).

Proposition 2.6. Let N and N′ be subgroups of a group G. Then every N-high subgroup of G is contained in some N′-high subgroup of G iff $N \cap N'$ is essential in N′.

Proof. It follows from 2.2.

The following Proposition 2.7 completes the Lemma 1.3.

Proposition 2.7. Let N and N′ be subgroups of a group. G. The following are equivalent:

(i) A subgroup A of G is N-high in G iff A is N′-high in G;

(ii) Every N-high subgroup of G is N′-high in G;

(iii) $N \cap N'$ is essential in N and in N′.

Proof. From 2.3 and 2.4.

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