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ON CONGRUENCES OF THE LATTICES SUB (L)

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Abstract: G. Grätzer suggested (see 1 , problem 7) to characterize the lattice Sub (L). In this paper some necessary conditions for L = Sub (L') for a finite lattice L are given.

Key words: Lattices Sub (L), transposes, f.c. elements, atomic congruence.

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1. Preliminaries. Let L be a lattice. Sub (L) denotes the lattice of all sublattices of L, ordered by the set inclusion. C(L) denotes the lattice of all congruences on L; ε denotes the smallest element of C(L) (identity on L).

The set consisting of elements a,b,... is denoted by (a,b,...). If M is a set, we sometimes write (M) instead of M. The symbols $\Cap$, $\Cup$ denote set intersection and union respectively.

The symbols $\cap$, $\cup$ denote the lattice operations of L. For the lattice operations of Sub (L) we use symbols $\wedge$, $\vee$. By $(a,b,\ldots)_L$, $(M)_L$ the sublattices of L generated by (a,b,...), M are denoted.

It is well-known that Sub (L) is a complete, atomic algebraic lattice having $\emptyset$ as the smallest element and L as the greatest one. All atoms in Sub (L) are precisely all one element subsets of L.

2. Congruences on Sub (L).

Definition. A lattice L is called subdirectly irreducible if for any arbitrary system $(\Theta_i)_I \subseteq C(L)$ holds: $\bigcap \Theta_i = \varepsilon$ implies that $\Theta_j = \varepsilon$ for some $j \in I$.

Definition. Two (closed) intervals $[a,b]$, $[c,d]$ are called transposes if $a = b \cap c$, $d = b \cup c$ or $c = a \cap d$, $b = a \cup d$. In the first case we write $[a,b] \nearrow [c,d]$, in the second case $[a,b] \searrow [c,d]$. It is obvious that the relations $\nearrow$, $\searrow$ are transitive.

We shall use the following lemmas.

Lemma 1 (see [1], p. 24). A reflexive and symmetric binary relation Θ on a lattice L is a congruence relation iff the following three properties are satisfied for all $x,y,z,t \in L$:

1) $x \Theta y$ iff $(x \cap y) \Theta (x \cup y)$.

2) $x \leq y \leq z$, $x \Theta y$, and $y \Theta z$ imply that $x \Theta z$.

3) $x \leq y$ and $x \Theta y$ imply that $(x \cup t) \Theta (y \cup t)$ and $(x \cap t) \Theta (y \cap t)$.

Lemma 2. Let L be a finite lattice, $\Theta \neq \varepsilon$ a congruence on Sub (L). Then there is an atom (a) in Sub (L), such that $(a) \equiv \emptyset\ (\Theta)$.

Lemma 3. If Θ is a congruence on a lattice L, $[p,q]$, $[r,s]$ are transposes, then $p \Theta q$ implies $r \Theta s$.

Definition. An element of a poset P is called fully comparable (f.c. element) if it is comparable with any element of P. A set consisting of f.c. elements is called f.c. set.

Remark. If $I \subseteq L$ is an f.c. set, $x \in \mathrm{Sub}(L)$, then $(x \uplus I) = (x \vee I) \in \mathrm{Sub}(L)$.

Definition. Let I be an f.c. subset of a lattice L. Let τ_I denote the binary relation on Sub (L) defined in the following way:

for $x, y \in \mathrm{Sub}(L)$ $x \, \tau_I y$ iff there is $J \subseteq I$ such that $(x \wedge y) \uplus J = x \vee y$.

It can be easily shown that τ_I is reflexive and symmetric.

Proposition 1. Let L be a lattice, I an f.c. subset of L. Then τ_I is a congruence relation on S b (L).

Proof. We shall verity the properties from Lemma 1.

1) Obvious.

2) $x \leq y \leq z$ and let $J, J' \subseteq I$ be such that $x \uplus J = (x \wedge y) \uplus J = x \vee y = y$ and $y \uplus J' = (y \wedge z) \uplus J' = y \vee z = z$. Then $z = x \uplus J \uplus J'$, i.e. $x \, \tau_I z$.

3) Let $x, y, t \in \mathrm{Sub}(L)$, $x \leq y$ and $x \, \tau_I y$. Then $x \uplus J = (x \wedge y) \uplus J = x \vee y = y$. But J is an f.c. subset (also a sublattice), so that $y \vee t = (x \uplus J) \vee t = (x \vee J) \vee t = x \vee t \vee J = (x \vee t) \uplus J$, thus $(x \vee t) \, \tau_I (y \vee t)$. Similarly $y \wedge t = (x \uplus J) \wedge t = (x \Cap t) \uplus (J \Cap t) = (x \wedge t) \uplus J'$, i.e. $(y \wedge t) \, \tau_I (x \wedge t)$.

The proof is finished.

The most important special case in the last definition is $I = (b)$. We shall write in this case τ_b instead of $\tau_{(b)}$.

Proposition 2. Let L be a finite lattice, $b \in L$ an f.c. element. Then the congruence τ_b is an atom in C(Sub (L)).

Proof. By Lemma 2 any congruence $\Theta \neq \varepsilon$ which is contained in τ_b, contains $[\emptyset, c]$ for some $c \in L$. By the definition of τ_b we have $c = b$, which implies that the congruence $\Theta \subset \tau_b$ is necessarily such that $(b) \equiv \emptyset(\Theta)$.

Further, by Lemma 3 and by definition, τ_b is the smallest of all congruences for which $(b) \equiv \emptyset$, so that $\Theta = \tau_b$.

Corollary 1. Let L be a finite lattice. Sub (L) is subdirectly irreducible iff card L = 1.

Proof. Since every finite lattice L, card $L \geq 2$ has $0 \neq 1$, the assertion follows from the last proposition.

Now, we shall convert Proposition 2 and show that any atomic congruence on Sub (L) has the form of τ_b for an f.c. element $b \in L$.

Theorem 1. Let L be a finite lattice. Then there is one-to-one correspondence between f.c. elements of L and atomic congruences on Sub (L); to an f.c. element b corresponds the congruence τ_b described above.

Proof. Let $\varphi \neq \varepsilon$ be a congruence on Sub (L). By Lemma 2 there is $b \in L$ such that $(b) \equiv \emptyset(\varphi)$. If b is an f.c. element, then by the proof of Proposition 2 $\tau_b \leq \varphi$ and we are done.

If b is not fully comparable, there is $c \in L$ such that $A = (b, c, b \cap c, b \cup c)$ is a four element sublattice of L. Since $(b) \wedge (c) = \emptyset$ and $(b) \vee (c) = A$ we have $[\emptyset, b] \nearrow [c, A]$ in Sub (L).

We show that $(b \cup c) \equiv \emptyset\ (\varphi)$.

Since $c \neq b \cup c$ we have $(c) \wedge (b \cup c) = \emptyset$ and $(c) \vee (b \cup c) = (c \uplus (b \cup c))_L = (c \uplus (b \cup c))$, thus $[\emptyset, b \cup c] \nearrow [c, c \uplus (b \cup c)]$.

Now $(b) \equiv \emptyset\ (\varphi)$ and $[\emptyset, b] \nearrow [c, A]$, hence $c \equiv A\ (\varphi)$.

Since $(c) \subset (c \uplus (b \cup c)) \subset A$ and every congruence class is a convex sublattice, we obtain

$c \equiv (c \uplus (b \cup c))$ $(\wp)$ and thus, finally

$(b \cup c) \equiv \emptyset$ $(\wp)$.

We shall distinguish two cases.

Case I. If $b \cup c$ is an f.c. element, then the congruence $\tau_{b \cup c}$ is an atom in C(Sub (L)), and since $b \neq b \cup c$, it is necessarily $\tau_{b \cup c} \subset \wp$. The proof is in this case finished.

Case II. If $b \cup c$ is not comparable with $d \in L$ we repeat the previous consideration and obtain $(b \cup c \cup d) \equiv \emptyset$ $(\wp)$. If $b \cup c \cup d$ is an f.c. element of L, we are finished. If not, we continue analogously. Since L is finite, we finally obtain an f.c. element $k \in L$ such that the atomic congruence τ_K is contained in $\wp$. The proof is finished.

Now we can describe a certain sublattice of C(Sub (L)) by using the atoms τ_b.

Theorem 2. Let L be a finite lattice, I the set of all f.c. elements of L, $(b_1, \dots, b_m) = J \subseteq I$. Then in C(Sub (L))

$$\tau_{b_1} \cup \tau_{b_2} \cup \dots \cup \tau_{b_m} = \tau_J$$

Proof. We denote the congruence on the left hand side by Θ. Let Φ be a congruence such that $\Phi \supseteq \tau_{b_i}$ for $i = 1,2,\dots,m$. Let $x \tau_J y$. Then there is $J' \subseteq J$ such that $x \vee y = (x \wedge y) \uplus J'$. Suppose $J' = (b_{k_1}, \dots b_{k_\ell})$, let $c_0 = x \wedge y$ and define $c_s = c_{s-1} \cup b_{k_s}$, $s = 1,2,\dots,\ell$. Then $c_0 < c_1 < \dots < c_\ell$ and $(c_{s-1}, c_s) \in \tau_{b_s} \leq \Phi$. Thus by the transitivity of Φ

$c_0 = (x \wedge y) \Phi (x \vee y) = c_\ell$. By Lemma 1 $x \Phi y$; thus $\tau_J \subseteq \Phi$ and, consequently $\tau_J = \Theta$.

Corollary 2. Let L be a finite lattice, I the set of all f.c. elements of L, card L = n. Then C(Sub (L)) contains as a sublattice the Boolean lattice 2^n having ε as the smallest element and τ_I as the greatest one.

Now, it is easy to reformulate our results as necessary conditions for a finite lattice L to be L = Sub (L′).

Let I denote the set of atoms of L the union of which with any different atom is of height 2. Let card I = n.

1) L is subdirectly reducible or card L = 2.

2) All atoms in C(L) are exactly the congruences τ_b (where $b \in I$) defined by $b \equiv 0$. The element b is the only element of L identified with 0 by the congruence τ_b.

3) The lattice C(L) contains a Boolean lattice 2^n as its sublattice. This lattice has ε as the smallest element and τ_b; $b \in I$ are exactly all its atoms.

R e f e r e n c e

[1] G. GRÄTZER: Lattice theory: First concepts and distributive lattices, Freeman, San Francisco 1971.

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