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AN ABELIAN ERGODIC THEOREM

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Abstract: An individual Abelian ergodic theorem is proved for a linear operator T on L_1 of a σ-finite measure space which satisfies certain boundedness conditions.

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Introduction. Derriennic and Lin ([3]) showed by an example that given an $\varepsilon > 0$ there exist a positive linear operator T on L_1 of a finite measure space, with T1 = 1 and $\| T^n \|_1 = 1 + \varepsilon$ for all $n \geq 1$, and a function f in L_1 such that the individual ergodic limit

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

does not exist almost everywhere on a certain measurable subset of positive measure. On the other hand, the author ([7]) has recently proved the following ergodic theorem.

Theorem A: Let T be a bounded linear operator on L_1 of a finite measure space and τ its linear modulus in the sense of Chacon and Krengel ([2]). Assume the conditions:

$$\sup_n \| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \|_1 < \infty \quad \text{and} \quad \sup_n \| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \|_\infty < \infty .$$

Then, for every f in L_∞, the ergodic limit

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

exists and is finite almost everywhere.

In connection with these results, it would be natural to ask whether the almost everywhere existence of the limit in Theorem A holds for every f in L_p with $1 < p < \infty$. Unfortunately, we do not know the answer even for T positive and power bounded with T1 = 1 (see also [31]). And this is the starting point for the work in this paper.

It will be observed below that if T is a bounded linear operator on L_1 of a σ-finite measure space such that $\sup_n \| \frac{1}{n} \sum_{i=0}^{n-1} T^i \|_\infty < \infty$ and also such that the adjoint of the linear modulus τ of T has a strictly positive subinvariant function s in L_∞, then for every $1 \le p < \infty$ and every f in L_p, the Abelian ergodic limit

$$\lim_{\lambda \to 1-0} (1 - \lambda) \sum_{n=0}^{\infty} \lambda^n T^n f(x)$$

exists and is finite almost everywhere.

<u>Abelian ergodic theorem</u>. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and $L_p(\mu) = L_p(X, \mathcal{F}, \mu)$, $1 \le p \le \infty$, the usual (complex) Banach spaces. Let T be a bounded linear operator on $L_1(\mu)$ and τ its linear modulus. T^* and τ^* will denote the corresponding adjoint operators on $L_1(\mu)^* = L_\infty(\mu)$. The following conditions (I) and (II) are assumed throughout the remainder of the paper:

(I) For some constant $K \geq 1$,

$$\sup_n \| \frac{1}{n} \sum_{i=0}^{n-1} T^i f \|_\infty \leq K \| f \|_\infty \quad \text{for all } f \in L_1(\mu) \cap L_\infty(\mu).$$

(II) There exists a function s in $L_\infty(\mu)$ satisfying

$$X = \{x : s(x) > 0\} \quad \text{and} \quad \tau^* s \leq s.$$

(We recall that T is a contraction, i.e., $\| T \|_1 \leq 1$ if and only if $\tau^* 1 \leq 1$, and that if τ satisfies

$$\sup_n \| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \|_1 < \infty \quad \text{and} \quad \lim_n \sup \| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i f \|_1 > 0$$

for every nonnegative f in $L_1(\mu)$ with $\| f \|_1 > 0$, then there exists a function s in L_∞ with $s > 0$ almost everywhere on X and $\tau^* s = s$ (cf. Corollary 2 of [61]).

Since $\int |Tf| \, s \, d\mu \leq \int (\tau |f|) s \, d\mu = \int |f| \tau^* s \, d\mu \leq \int |f| s \, d\mu$ for all $f \in L_1(\mu)$, and since $L_1(\mu)$ is a dense subspace of $L_1(s \, d\mu) = L_1(X, \mathcal{F}, s \, d\mu)$, T may be regarded as a linear contraction operator on $L_1(s \, d\mu)$. Clearly, T on $L_1(s \, d\mu)$ satisfies

$$\sup_n \| \frac{1}{n} \sum_{i=0}^{n-1} T^i f \|_\infty \leq K \| f \|_\infty \quad \text{for all } f \in L_1(s \, d\mu) \cap L_\infty(s \, d\mu).$$

Therefore, by the Riesz convexity theorem, T also may be regarded as a linear operator on each $L_p(s \, d\mu)$, with $1 \leq p < \infty$, such that

$$\sup_n \| \frac{1}{n} \sum_{i=0}^{n-1} T^i \|_p \leq K.$$

It then follows that $\sup_n \| (1/n) T^n \|_p < \infty$, and hence $\lim_n \| T^n \|_p^{1/n} \leq 1$. Thus, for every $0 < \lambda < 1$, $\sum_{n=0}^{\infty} \lambda^n T^n$ is a bounded linear operator on $L_p(s \, d\mu)$, and it also follows that, for every $f \in L_p(s \, d\mu)$, $\sum_{n=0}^{\infty} \lambda^n |T^n f(x)| < \infty$ for almost all $x \in X$.

Under these circumstances, we shall prove the following theorem.

<u>Theorem</u>: For every $1 \leq p < \infty$ and every $f \in L_p(s\, d\mu)$, the limit

$$\lim_{\lambda \to 1-0} (1-\lambda) \sum_{n=0}^{\infty} \lambda^n T^n f(x)$$

exists and is finite for almost all $x \in X$.

For the proof of this theorem, we need two lemmas. The first one is a slight generalization of Chacon's maximal ergodic lemma ([1]).

<u>Lemma 1</u>: For every $1 \leq p < \infty$, every $f \in L_p(s\, d\mu)$ and every constant $a > 0$, we have

$$\int_{\{f^* > K^2 a\}} (a - \min\{|f(x)|, a\})\, d\mu \leq \int_{\{|f| > a\}} (|f(x)| - a)\, d\mu,$$

where f^* is defined by

$$f^*(x) = \sup_n \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) \right| \qquad (x \in X).$$

<u>Proof</u>: Since Chacon's argument ([1]) can be easily modified to yield a proof of this lemma, we omit the details.

<u>Lemma 2</u>: For every $1 \leq p < \infty$ and every $f \in L_p(s\, d\mu)$, let

$$\bar{f}(x) = \sup_{0 < \lambda < 1} \left| (1-\lambda) \sum_{n=0}^{\infty} \lambda^n T^n f(x) \right| \qquad (x \in X).$$

Then $\bar{f}(x) < \infty$ for almost all $x \in X$.

<u>Proof</u>: Since there exists a μ-null set N such that if $x \notin N$ then

$$\sum_{n=0}^{\infty} \lambda^n |T^n f(x)| < \infty \quad \text{for all } 0 < \lambda < 1,$$

we get, for all $x \notin N$,

$$(1-\lambda)\sum_{n=0}^{\infty} \lambda^n T^n f(x) = (1-\lambda)^2 \sum_{n=0}^{\infty} [\lambda^n \sum_{i=0}^{n} T^i f(x)]$$

$$= (1-\lambda)^2 \sum_{n=0}^{\infty} [(n+1)\lambda^n (\frac{1}{n+1}\sum_{i=0}^{n} T^i f(x))] .$$

Since $(1-\lambda)^2 \sum_{n=0}^{\infty}(n+1)\lambda^n = 1$, it follows that $\overline{f}(x) \le f^*(x)$ for all $x \notin N$. Therefore it suffices to show that $f^*(x) < \infty$ for almost all $x \in X$.

To do this, we apply Lemma 1 and obtain, for every $a > 0$,

$$\frac{a}{2}\mu(\{f^* > K^2 a\} - \{|f| > \frac{a}{2}\}) \le \int_{\{f^* > K^2 a\}} (a - \min\{|f(x)|, a\})d\mu$$

$$\le \int_{\{|f| > a\}} (|f(x)| - a)\, d\mu .$$

Thus, for every $a > 0$, we have

$$\frac{a}{2}\mu(\{f^* > K^2 a\}) \le \frac{a}{2}\mu(\{|f| > \frac{a}{2}\}) + \int_{\{|f| > a\}} (|f(x)| - a)d\mu$$

$$\le \int_{\{|f| > \frac{a}{2}\}} |f(x)|\, d\mu ,$$

and so, letting $a \longrightarrow \infty$, the desired conclusion follows.

<u>Proof of the Theorem</u>: For $1 < p < \infty$, $L_p(s\, d\mu)$ is a reflexive Banach space. Then, since $\sup_n \| \frac{1}{n}\sum_{i=0}^{n-1} T^i \|_p \le K$ and

$$\lim_n \|(1/n)T^n f\|_p^p \le (\sup \|(1/n)T^n f\|_\infty)^{p-1} \lim_n \|(1/n)T^n f\|_1 = 0$$

for all $f \in L_1(s\, d\mu) \cap L_\infty(s\, d\mu)$, it follows (cf. Corollaries 5.2 and 5.4 in Chapter VIII of [4]) that the set

$$L = \{g - Tg + h : g, h \in L_p(s\, d\mu) \text{ and } Th = h\}$$

is dense in $L_p(s\, d\mu)$.

We notice that if $f \in L$, then the ergodic limit in the

Theorem exists and is finite for almost all $x \in X$. In fact, this follows from considering the case $f = g - Tg$, with $g \in L_p(s\, d\mu)$. If this is the case, then we have for almost all $x \in X$,

$$\left|(1-\lambda)\sum_{n=0}^{\infty} \lambda^n T^n f(x)\right| \leq (1-\lambda)(|g(x)| + \overline{Tg}(x)).$$

Hence, by Lemma 2, $\lim_{\lambda \to 1-0} (1-\lambda)\sum_{n=0}^{\infty} \lambda^n T^n f(x) = 0$ for almost all $x \in X$.

By this and Lemma 2, we can apply Banach's convergence theorem ([4], p. 332) to infer that, for every $f \in L_p(s\, d\mu)$ with $1 < p < \infty$, the ergodic limit in the Theorem exists and is finite for almost all $x \in X$. Since $L_1(s\, d\mu) \cap L_p(s\, d\mu)$ is dense in $L_1(s\, d\mu)$, we can apply Lemma 2 and Banach's convergence theorem again to infer that, for every $f \in L_1(s\, d\mu)$, the ergodic limit in the Theorem exists and is finite for almost all $x \in X$.

The proof is complete.

If we assume, in addition, that T is positive, then we can apply the Chacon-Ornstein lemma ([5], p. 22) and obtain that, for every $f \in L_1(s\, d\mu)$, $\lim_n (1/n)T^n f(x) = 0$ for almost all $x \in X$. Therefore the above argument shows that, for every $1 \leq p < \infty$ and every $f \in L_p(s\, d\mu)$, the limit

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

exists and is finite for almost all $x \in X$.

Although we do not know whether this result holds without assuming that T is positive, the next proposition gives a partial answer.

Proposition: If X is countable, then for every $1 \leq p < \infty$ and every $f \in L_p(s\, d\mu)$, the limit

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

exists and is finite for almost all $x \in X$.

Proof: Without loss of generality we may assume that $0 < \mu(\{x\}) < \infty$ for each $x \in X$. Let (k_n) be any strictly increasing sequence of positive integers, and take a subsequence (j_n) of (k_n) so that

$$\sum_{n=1}^{\infty} (1/j_n) < \infty .$$

Then, for all $f \in L_1(s\, d\mu)$, we have

$$\sum_{n=1}^{\infty} (1/j_n) \| T^{j_n} f \|_1 < \infty ,$$

and hence

$$\lim_n (1/j_n) T^{j_n} f(x) = 0$$

for all $x \in X$. This and the argument used in the proof of the Theorem imply that, for every $1 \leq p < \infty$ and every $f \in L_p(s\, d\mu)$, the limit

$$\lim_n \frac{1}{j_n} \sum_{i=0}^{j_n - 1} T^i f(x)$$

exists and is finite for all $x \in X$. We have now proved that every strictly increasing sequence (k_n) of positive integers has a subsequence (j_n) such that, for every $1 \leq p < \infty$ and every $f \in L_p(s\, d\mu)$, the limit

$$\lim_n \frac{1}{j_n} \sum_{i=0}^{j_n - 1} T^i f(x)$$

exists and is finite for all $x \in X$.

Hence the Proposition follows from the mean ergodic theorem for $1 < p < \infty$.

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