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ON A CLASS OF TANGENTIAL CAUCHY-RIEMANN MAPS

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Abstract: We present conditions for a tangential Cauchy-Riemann map $f: S^3 \longrightarrow \mathbb{H}^1$, $S^3 \subset \mathbb{H}^2$ a unit hypersphere and $\mathbb{H}^N$ a Hermitian space, to be constant.

Key words: Hermitian space, tangential Cauchy-Riemann map, integral formula, harmonic map.

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Be given a Hermitian plane $\mathbb{H}^2$, i.e., the Euclidean space E^4 together with an endomorphism $J: V(E^4) \longrightarrow V(E^4)$, $V(E^4)$ being the vector space of E^4, such that $J^2 = -\text{id.}$, $\langle Ju, Jv \rangle = \langle u, v \rangle$ for each $u, v \in V(E^4)$. By a frame of $\mathbb{H}^2$ we mean each frame $\varphi = \{m; v_1, v_2, v_3, v_4\}$ of E^4 such that $v_2 = Jv_1$, $v_4 = Jv_3$. Let φ be a field of frames of $\mathbb{H}^2$; then

(1)
$$\begin{aligned} dm &= \omega^1 v_1 + \omega^2 v_2 + \omega^3 v_3 + \omega^4 v_4, \\ dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4, \\ dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4, \\ dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4, \\ dv_4 &= -\omega_1^3 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3 \end{aligned}$$

together with the integrability conditions $(i,j,k = 1,\dots,4)$

(2)
$$d\omega^i = \omega^j \wedge \omega_j^i, \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j.$$

From $dv_2 = Jdv_1$, $dv_4 = Jdv_3$, we get

(3) $$\omega_1^3 = \omega_2^4, \quad \omega_2^3 = -\omega_1^4.$$

Let $S^3 \subset \mathbb{H}^2$ be a unit hypersphere. Let us associate to each of its points m a frame φ in such a way that $m + v_4$ is its center; evidently, we get

(4) $$\omega^4 = 0, \ \omega_1^4 = \omega^1, \ \omega_2^4 = \omega^2, \ \omega_3^4 = \omega^3.$$

Further, let $\mathbb{H}^1$ be a Hermitian straight line, i.e., the Euclidean plane E^2 endowed by an endomorphism $\mathcal{J} : V(E^2) \rightarrow V(E^2)$, $\mathcal{J}^2 = -\text{id.}$, $\langle \mathcal{J}x, \mathcal{J}y \rangle = \langle x,y \rangle$ for $x,y \in V(E^2)$. For its frames $\sigma = \{n; w_1, w_2\}$, $w_2 = \mathcal{J}w_1$, we may write

(5) $$dn = \varphi^1 w_1 + \varphi^2 w_2, \ dw_1 = \varphi_1^2 w_2, \ dw_2 = -\varphi_1^2 w_1;$$

(6) $$d\varphi^1 = -\varphi^2 \wedge \varphi_1^2, \ d\varphi^2 = \varphi^1 \wedge \varphi_1^2, \ d\varphi_1^2 = 0.$$

Be given a mapping $f: S^3 \longrightarrow \mathbb{H}^1$. Let us write $\tau^1 := f^* \varphi^1$, $\tau^2 := f^* \varphi^2$, $\tau_1^2 := f^* \varphi_1^2$. Then

(7) $$\tau^1 = a_1^1\omega^1 + a_2^1\omega^2 + a_3^1\omega^3, \quad \tau^2 = a_1^2\omega^1 + a_2^2\omega^2 + a_3^2\omega^3$$

and the differential $f_* : T_m(S^3) \longrightarrow T_{f(m)}(\mathbb{H}^1)$, $m \in S^3$, is given by

(8) $$f_* v_1 = a_1^1 w_1 + a_1^2 w_2.$$

Let $\tau_m \subset T_m(S^3)$ be the plane given by $J\tau_m = \tau_m$; τ_m is spanned by v_1 and v_2. The mapping f is said to satisfy the <u>tangential Cauchy-Riemann condition</u>, see [2], if $\mathcal{J} \circ f_* = f_* \circ J$; let us call it TCR. It is easy to see that f is TCR if and only if

$$(9) \qquad a_1^1 - a_2^2 = 0, \quad a_1^2 + a_2^1 = 0,$$

i.e., (7) are of the form

$$(10) \qquad \tau^1 = a\omega^1 - b\omega^2 + c\omega^3, \quad \tau^2 = b\omega^1 + a\omega^2 + e\omega^3.$$

By the exterior differentiation,

$$(11) \quad \{da + b(\omega_1^2 - \tau_1^2)\} \wedge \omega^1 - \{db - a(\omega_1^2 - \tau_1^2)\} \wedge \omega^2 + (dc - e\tau_1^2) \wedge \omega^3 = -2c\omega^1 \wedge \omega^2 - b\omega^1 \wedge \omega^3 - a\omega^2 \wedge \omega^3,$$

$$\{db - a(\omega_1^2 - \tau_1^2)\} \wedge \omega^1 + \{da + b(\omega_1^2 - \tau_1^2)\} \wedge \omega^2 + (de + c\tau_1^2) \wedge \omega^3 = -2e\omega^1 \wedge \omega^2 + a\omega^1 \wedge \omega^3 - b\omega^2 \wedge \omega^3,$$

and we get the existence of functions $A, \dots, F: S^3 \longrightarrow \mathbb{R}$ such that

$$(12) \qquad \begin{aligned} da + b(\omega_1^2 - \tau_1^2) &= (A - e)\omega^1 + (B + c)\omega^2 + C\omega^3, \\ db - a(\omega_1^2 - \tau_1^2) &= (c - B)\omega^1 + (A + e)\omega^2 + D\omega^3, \\ dc - e\tau_1^2 &= (C - b)\omega^1 - (D + a)\omega^2 + E\omega^3, \\ de + c\tau_1^2 &= (D + a)\omega^1 + (C - b)\omega^2 + F\omega^3. \end{aligned}$$

Further,

$$(13) \quad \{dA + B(\tau_1^2 - 2\omega_1^2)\} \wedge \omega^1 + \{dB - A(\tau_1^2 - 2\omega_1^2)\} \wedge \omega^2 + \{dC + D(\omega_1^2 - \tau_1^2)\} \wedge \omega^3 = - 4C\omega^1 \wedge \omega^2 + (B - F + c)\omega^1 \wedge \omega^3 + (E - A + e)\omega^2 \wedge \omega^3,$$

$$- \{dB - A(\tau_1^2 - 2\omega_1^2)\} \wedge \omega^1 + \{dA + B(\tau_1^2 - 2\omega_1^2)\} \wedge$$

$$\wedge\omega^2 + \{dD - C(\omega_1^2 - \tau_1^2)\}\wedge\omega^3 =$$

$$= -4D\omega^1\wedge\omega^2 + (A + E + e)\omega^1\wedge\omega^3 + (B + F - c)\omega^2\wedge$$

$$\wedge\omega^3,$$

$$\{dC + D(\omega_1^2 - \tau_1^2)\}\wedge\omega^1 - \{dD - C(\omega_1^2 - \tau_1^2\}\wedge\omega^2 +$$

$$+ (dE - F\tau_1^2)\wedge\omega^3 =$$

$$= -2(E + e)\omega^1\wedge\omega^2 - (2D + a)\omega^1\wedge\omega^3 - (2C - b)\omega^2\wedge$$

$$\wedge\omega^3,$$

$$\{dD - C(\omega_1^2 - \tau_1^2)\}\wedge\omega^1 + \{dC + D(\omega_1^2 - \tau_1^2)\}\wedge\omega^2 +$$

$$+ (dF + E\tau_1^2)\wedge\omega^3 =$$

$$= -2(F - c)\omega^1\wedge\omega^2 + (2C - b)\omega^1\wedge\omega^3 - (2D + a)\omega^2\wedge$$

$$\wedge\omega^3,$$

this implying the existence of functions $K,\dots,S\colon S^3 \to \mathbf{R}$ satisfying

(14) $$dA + B(\tau_1^2 - 2\omega_1^2) = (L - 2D)\omega^1 - (K - 2C)\omega^2 + M\omega^3,$$

$$dB - A(\tau_1^2 - 2\omega_1^2) = -(K + 2C)\omega^1 - (L + 2D)\omega^2 + N\omega^3,$$

$$dC + D(\omega_1^2 - \tau_1^2) = (M + B - F + c)\omega^1 + (N - A + E + e)\omega^2 + P\omega^3,$$

$$dD - C(\omega_1^2 - \tau_1^2) = -(N - A - E - e)\omega^1 + (M + B + F - c)\omega^2 + Q\omega^3,$$

$$dE - F\tau_1^2 = (P - 2D - a)\omega^1 - (Q + 2C - b)\omega^2 + R\omega^3,$$

$$dF + E\tau_1^2 = (Q + 2C - b)\omega^1 + (P - 2D - a)\omega^2 + S\omega^3.$$

Let $\tilde{\varphi} = \{m; \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4\}$, $\tilde{\sigma} = \{n; \tilde{w}_1, \tilde{w}_2\}$ be another fields of frames, let

(15) $v_1 = \cos\alpha.\tilde{v}_1 - \sin\alpha.\tilde{v}_2$, $v_2 = \sin\alpha.\tilde{v}_1 + \cos\alpha.\tilde{v}_2$,

$v_3 = \tilde{v}_3$, $v_4 = \tilde{v}_4$;

$w_1 = \cos\beta.\tilde{w}_1 - \sin\beta.\tilde{w}_2$, $w_2 = \sin\beta.\tilde{w}_1 + \cos\beta.\tilde{w}_2$.

Then the direct calculation yields

(16) $\tilde{\omega}^1 = \cos\alpha.\omega^1 + \sin\alpha.\omega^2$, $\tilde{\omega}^2 = -\sin\alpha.\omega^1 + \cos\alpha.\omega^2$, $\tilde{\omega}^3 = \omega^3$, $\tilde{\omega}^2_1 = \omega^2_1 + d\alpha$;

$\tilde{\varphi}^1 = \cos\beta.\omega^1 + \sin\beta.\omega^2$, $\tilde{\varphi}^2 = -\sin\beta.\varphi^1 + \cos\beta.\varphi^2$;

(17) $\tilde{a} = \cos(\alpha-\beta).a - \sin(\alpha-\beta).b$, $\tilde{b} = \sin(\alpha-\beta).a + \cos(\alpha-\beta).b$,

$\tilde{c} = \cos\beta.c + \sin\beta.e$, $\tilde{e} = -\sin\beta.c + \cos\beta.e$;

(18) $\tilde{A} = \cos(2\alpha-\beta).A + \sin(2\alpha-\beta).B$, $\tilde{B} = -\sin(2\alpha-\beta).A + \cos(2\alpha-\beta).B$,

$\tilde{C} = \cos(\alpha-\beta).C - \sin(\alpha-\beta).D$, $\tilde{D} = \sin(\alpha-\beta).C + \cos(\alpha-\beta).D$,

$\tilde{E} = \cos\beta.E + \sin\beta.F$, $\tilde{F} = -\sin\beta.E + \cos\beta.F$.

Lemma. The 2-form

(19) $\Omega = 2(aC + bD)\omega^1\wedge\omega^2 - \{a(2B - F) + b(2A - E) + eC - cD - ac - be\}\omega^1\wedge\omega^3 + \{a(2A - E) - b(2B + F) + cC + eD + ae - bc\}\omega^2\wedge\omega^3$

on S^3 is invariant and we have

(20) $d\Omega = 4(a^2 + b^2 + A^2 + B^2 + C^2 + D^2 + eE - cF)\, dv$,

$dv = \omega^1 \wedge \omega^2 \wedge \omega^3$ being the volume element of S^3. The function

(21) $$T = eE - cF$$

is invariant as well.

Theorem 1. Be given a TCR mapping $f: S^3 \longrightarrow \mathbb{H}^1$ satisfying $T \geq 0$. Then f is a constant mapping.

Proof. From the integral formula $\int_{S^3} d\Omega = 0$, we get $a = b = A = B = C = D = 0$. From (12_1), $0 = -e\omega^1 + c\omega^2$, i.e., $c = e = 0$ and $\tau^1 = \tau^2 = 0$. QED.

The harmonicity of a TCR mapping $f: S^3 \longrightarrow \mathbb{H}^1$ should imply its constancy. Let us now prove a generalization of this assertion. Be given an arbitrary mapping $f: S^3 \longrightarrow \mathbb{H}^1$,

(22) $$\tau^\alpha = a_i^\alpha \omega^i \quad (\alpha = 1,2).$$

From this,

(23) $$(da_i^\alpha - a_j^\alpha \omega_i^j + a_i^\beta \tau_\beta^\alpha) \wedge \omega^i,$$

and we get the existence of functions $a_{ij}^\alpha = a_{ji}^\alpha : S^3 \longrightarrow \mathbb{R}$ such that

(24) $$da_i^\alpha - a_j^\alpha \omega_i^j + a_i^\beta \tau_\beta^\alpha = a_{ij}^\alpha \omega^j.$$

To each point $m \in S^3$, let us associate the vector

(25) $$t = \delta^{ij} a_{ij}^\alpha w_\alpha \in T_{f(m)}(\mathbb{H}^1);$$

the mapping t is called the <u>tension field</u>, see [1] or [3] resp. The mapping f is called <u>harmonic</u> if $t = 0$ on S^3. Now, let f be our TCR map. From (24) and (12), it is easy to see that

(26) $a_{11}^1 = A - e$, $a_{22}^1 = -A - e$, $a_{33}^1 = E$, $a_{12}^1 = B$, $a_{13}^1 = C$,

$a^1_{23} = -D$,

$a^2_{11} = -B + c$, $a^2_{22} = B + c$, $a^2_{33} = F$, $a^2_{12} = A$, $a^2_{13} = D$,

$a^2_{23} = C$,

i.e.,

(27) $$t = (E - 2e)w_1 + (F + 2c)w_2.$$

For each $m \in S^3$, consider the vectors

(28) $$t_1 = f_* v_3 = cw_1 + ew_2,\ t_2 = \mathcal{J}(f_* v_3) = -ew_1 + cw_2,$$
$$t_3 = v_3(f_* v_3) = Ew_1 + Fw_2.$$

Then

(29) $$t = t_3 + 2t_2$$

and

(30) $$\langle t_2, t_3 \rangle = -T,$$

this presenting the geometrical description of T.

<u>Theorem 2</u>. Let $f: S^3 \longrightarrow H^1$ be a TCR map and let there exist functions $p,q: S^3 \longrightarrow \mathbb{R}$, $q \geq 0$ on S^3, such that

(31) $$t_3 + pt_1 + qt_2 = 0.$$

Then f is a constant mapping.

<u>Proof</u>. All we have to do is to prove $T \geq 0$. From (31), $E + pc - qe = F + pe + qc = 0$, $T = q(e^2 + c^2)$, and we are done.

References

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