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ON THE STRICT CONVEXITY OF THE POLAR OPERATOR

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Abstract: There is proved that the polar operator is convex in any linear topological space and strictly convex in any separated locally convex space.

Key words: Linear topological spaces, locally convex spaces, polar operator, convexity, strict convexity.

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The purpose of this note is to prove the following theorem.

Theorem. Let X be a separated real locally convex space, $n \geq 1$ an integer, $A_1, \ldots, A_n$ nonempty subsets of X and $t_1, \ldots, t_n$ nonnegative numbers with $\sum_{i=1}^{n} t_i = 1$. Then

$$(\sum_{i=1}^{n} t_i A_i)^{o} \subset (\sum_{i=1}^{n} t_i A_i^{o})_{*} .$$

The equality holds if and only if $\operatorname{cocl}(A_i \cup \{0\}) = \operatorname{cocl}(A_j \cup \{0\})$ for all i, j with $t_i t_j > 0$.

If X is a real linear topological space, M a subset of X and N a subset of the dual space X', then $\operatorname{co}(M)$, $\operatorname{cl}(M)$, $\operatorname{cocl}(M)$ denotes the convex hull, closure and convex closed hull of M, respectively, and $M^{o} = \{x' \in X' : \langle M, x' \rangle \leq 1\}$, $N^{o} = \{x \in X : \langle x, N \rangle \leq 1\}$ the polar sets of M and N, respectively (where, for example, $\langle M, x' \rangle \leq 1$ means that one is an upper bound for the set $\langle M,x' \rangle = \{\langle x,x' \rangle : x \in M\}$).

If X is a linear space and M a subset of X, then M_* denotes the set $\bigcup \{[0,x] : [0,x) \subset M\} = \{x \in X : [0,x) \subset M\}$ ($[0,0) = \{0\}$).

V.P. Fedotov [1] asserts that if $A_1,\dots,A_n$ are closed convex subsets of a real separated locally convex space X containing the origin, then

$$\frac{A_1 + \dots + A_n}{n} \supset \left(\frac{A_1^o + \dots + A_n^o}{n} \right)^o ,$$

the equality being true iff $A_1 = \dots = A_n$. It seems that his consideration implies only that this inequality holds if the left hand side of it is replaced by its closure (or by its $(.)_*$ -closure).

Our lemma 3 almost coincides with [1, Lemma 1]. Lemma 4 below has been indicated by Fedotov in [1, Lemma 2] in case $t = \frac{1}{2}$ but his proof is not clear (it seems that it contains a gap at the induction step and that a lemma like our lemma 2 is necessary).

In what follows our Theorem is divided into two theorems 1 and 2. The proof of Theorem 2 is quite different from that of the corresponding part of [1] and seems to be more straightforward.

The proof of the following easy lemma is omitted.

<u>Lemma 1</u>. Let X be a linear space and M a subset of X. Then the following assertions hold:

(i) $M_* \neq \emptyset$ if and only if M contains 0;

(ii) $M \subset M_*$ if and only if M is starshaped (relative to 0);

(iii) $M_* = \bigcap_{r>0} (1 + r)M$ whenever M contains 0;

(iv) If C is a linear topological space, then $M_* \subset$ $\subset cl(M)$;

(v) if X is a linear topological space and M is star-shaped (relative to 0), then $M \subset M_* \subset cl(M)$.

<u>Theorem 1</u>. Let X be a (possibly non-separated) real linear topological space, $n \geq 1$ an integer, $A_1,\dots,A_n$ nonempty subsets of X and $t_1,\dots,t_n$ nonnegative numbers with $\sum_{i=1}^n t_i = 1$. Then

$$(\sum_{i=1}^n t_i A_i)^0 \subset (\sum_{i=1}^n t_i A_i^0)_* \quad .$$

<u>Proof</u>. We may restrict ourselves to the case when all A_i's are convex and contain 0. Let x' in $(\sum_{i=1}^n t_i A_i)^0$ be given and set $h_i = \sup \langle A_i, x' \rangle \in [0,+\infty]$. Then $h_i^{-1} x' \in A_i^0$ $(\infty^{-1} = 0)$ whenever $h_i > 0$, so that

$$(1) \qquad (\sum_+ t_i h_i^{-1}) x' \in \sum_+ t_i A_i^0$$

where $\sum_+$ is the summation over all i's with $h_i > 0$. If $h_i =$ $= 0$ and $a > 0$, then $a^{-1}x' \in A_i^0$ so that

$$(2) \qquad (\sum_0 t_i a^{-1}) x' \in \sum_0 t_i A_i^0,$$

where $\sum_0$ denotes the summation over all i's with $h_i = 0$. From (1) and (2) it follows that

$$(3) \qquad (\sum_+ t_i h_i^{-1} + \sum_0 t_i a^{-1}) x' \in \sum_{i=1}^n t_i A_i^0$$

for each $a > 0$.

If $t_i > 0$, then h_i is finite, because $t_i A_i \subset \sum_{i=1}^n t_i A_i$ implies $x' \in (t_i A_i)^0 = t_i^{-1} A_i^0$ so that $h_i = \sup \langle A_i, x' \rangle =$ $= t_i^{-1} \sup \langle t_i A_i, x' \rangle \leq t_i^{-1}$. Hence $h_i = +\infty$ implies $t_i = 0$.

Let $b \in (0,+\infty)$ be arbitrary and set $g_i = h_i$ if h_i is finite and $g_i = b$ otherwise. Then, by the Cauchy-Schwarz inequality,

$$(\textstyle\sum_+ t_i g_i^{-1} + \sum_0 t_i a^{-1})(\sum_+ t_i g_i + \sum_0 t_i a) \geq$$

$$\geq (\textstyle\sum_+ t_i g_i^{-1} g_i + \sum_0 t_i a_i^{-1} a)^2 = (\sum_{i=1}^n t_i)^2 = 1.$$

Letting $b \to +\infty$, we see that

$$(\textstyle\sum_+ t_i h_i^{-1} + \sum_0 t_i a^{-1})(\sum_+ t_i h_i + \sum_0 t_i a) \geq 1,$$

if we agree that $t_i h_i = 0$ whenever $h_i = +\infty$ (and, consequently, $t_i = 0$). From this and (3) it follows that

(4) $$(\textstyle\sum_+ t_i h_i + \sum_0 t_i a)^{-1} x' \in \sum_{i=1}^n t_i A_i^o .$$

It is easy to see that

$$\textstyle\sum_+ t_i h_i = \sum_+ t_i h_i + \sum_0 t_i h_i = \sup \langle \sum_{i=1}^n t_i A_i, x' \rangle \leq$$
$$\leq 1$$

so that $\sum_+ t_i h_i + \sum_0 t_i a \leq 1 + a$. Hence, by (4), $(1 + a)^{-1} x' \in \sum_{i=1}^n t_i A_i^o$, i.e., $x' \in (1 + a) \sum_{i=1}^n t_i A_i^o$ for each $a > 0$. By lemma 1, (iii), $x' \in (\sum_{i=1}^n t_i A_i^o)$.

The proof is completed.

__Lemma 2.__ Let $0 < t < 1$. Then the following definition (by induction) of two sequences $\{u_k\}_{k=0}^{\infty}$ and $\{v_k\}_{k=0}^{\infty}$ is correct:

(5)
$$u_0 = t, \qquad v_0 = 1 - t,$$
$$u_{k+1} = \frac{u_0}{1 - v_0 v_k} \qquad v_{k+1} = \frac{v_0}{1 - u_0 u_k} .$$

Moreover, both sequences lie in $(0,1)$, strictly increase and converge to one.

__Proof.__ We shall prove, by induction, the following assertion:

(6_n) $\{u_k\}_{k=o}^n$, $\{v_k\}_{k=o}^n$ are well defined and strictly increasing sequences contained in $(0,1)$.

(6_1) is true, because $1 > 1 - u_o^2 > 0$, $1 > 1 - v_o^2 > 0$, so that

$$1 > u_1 = \frac{u_o}{1 - v_o v_o} > u_o, \quad 1 > v_1 = \frac{v_o}{1 - u_o u_o} > v_o.$$

Suppose that (6_n) is true for some $n = m \geq 1$. Then we have

$$u_{m+1} - u_m = \frac{u_o v_o (v_m - v_{m-1})}{(1 - v_o v_m)(1 - v_o v_{m-1})}.$$

As $1 > v_m > v_{m-1} > 0$ and $u_m > 0$ (by the inductive hypothesis), we have $1 > 1 - v_o v_m > 0$, $1 > 1 - v_o v_{m-1} > 0$ and, consequently, $u_{m+1} > u_m$. The inequality $u_{m+1} < 1$ follows from

$$u_{m+1} = \frac{u_o}{1 - v_o v_m} = \frac{1 - v_o}{1 - v_o v_m} < \frac{1 - v_o v_m}{1 - v_o v_m} = 1.$$

Similarly $v_m < v_{m+1} < 1$. Hence (6_n) holds for each n.

Let $u = \lim u_k$, $v = \lim v_k$. From (5) it follows that

$$u = \frac{u_o}{1 - v_o v} \quad \text{and} \quad v = \frac{v_o}{1 - u_o u}$$

leading to the following equation for u:

$$u_o u^2 - (1 + u_o^2 - v_o^2)u + u_o = 0.$$

As $1 + u_o^2 - v_o^2 = 1 + u_o - v_o = 2u_o$, the last equation is of the form

$$u_o u^2 - 2u_o u + u_o = u_o(u - 1)^2 = 0.$$

This equation has the unique solution $u = 1$. Similarly $v = 1$.

Lemma 3. Let X be a separated locally convex space and A,B,C three nonempty subsets of X. If C absorbs A and $A + C \supset$

$\supset A + B$, then cocl $(C) \supset B$.

Proof. We may suppose that X is a real locally convex space. Let us suppose that there is a point x in B which is not in cocl (C). Then there exists x' in X' such that $\langle x,x' \rangle > \sup \langle C,x' \rangle$. As C absorbs A, the number $\sup \langle A,x' \rangle$ is finite. Then $\sup \langle A + C,x' \rangle = \sup \langle A,x' \rangle + \sup \langle C,x' \rangle < \sup \langle A,x' \rangle + \langle x,x' \rangle \leq \sup \langle A,x' \rangle + \sup \langle B,x' \rangle = \sup \langle A + B, x' \rangle \leq \sup \langle A + C,x' \rangle$, a contradiction.

Lemma 4. Let X be a real separated locally convex space, A,B, and C three nonempty subsets of X and $0 < t < 1$. If $tA + (1 - t)B \subset C$ and $tA^o + (1 - t)B^o \subset C^o$, then

$$\text{cocl } (A \cup \{0\}) = \text{cocl } (B \cup \{0\}) = \text{cocl } (C \cup \{0\}).$$

Proof. It is clear that we may restrict ourselves to the case when all sets A,B,C are convex, closed and contain 0, and to show that A = B = C.

Let $\{u_k\}_{k=0}^{\infty}$ and $\{v_k\}_{k=0}^{\infty}$ be the sequences from lemma 2. We shall show, by induction, that

$$(7_n) \qquad u_nA \subset C \subset u_n^{-1}A, \quad v_nB \subset C \subset v_n^{-1}B$$

holds for all $n \geq 0$.

(7_0) is true because $u_0A, v_0B \subset u_0A + v_0B \subset C$ and u_0A^o, $v_0B^o \subset u_0A^o + v_0B^o \subset C^o$, i.e. $C = C^{oo} \subset (u_0A^o)^o = u_0^{-1}A$, $(v_0B^o)^o = v_0^{-1}B$. Let (7_n) hold for some $n = m \geq 0$. Then

$$u_0A + v_0B \subset C = (1 - v_0v_m)C + v_0v_mC \subset (1 - v_0v_m)C + v_0B$$

so that, by lemma 3, $u_0A \subset (1 - v_0v_m)C$, i.e. $u_{m+1}A \subset C$. Similarly $v_{m+1}B \subset C$. The other two inclusions in (7_{m+1}) follow in the same manner by considering the polar sets to A,B, and C. Hence (7_n) holds for all $n \geq 0$.

As $u_nx \in C$ for each $n \geq 0$ and $x \in A$, and $u_n \longrightarrow 1$, we have that $A \subset C$. Similarly one sees that $C \subset A$ and $B \subset C \subset B$.

Theorem 2. Let the hypotheses of Theorem 1 be satisfied. If X is locally convex and $(\sum_{i=1}^{n} t_i A_i)^o = (\sum_{i=1}^{n} t_i A_i^o)_*$, then $\mathrm{cocl}\,(A_i \cup \{0\}) = \mathrm{cocl}\,(A_j \cup \{0\})$ for all i, j with $t_i t_j > 0$.

Proof. From lemma 1, (v) it follows that $(\sum_{i=1}^{n} t_i A_i^o)_* = \mathrm{cl}\,(\sum_{i=1}^{n} t_i A_i^o)$. It is clear that we may restrict ourselves to the case when $n > 1$, all A_i's are convex, closed and contain 0, and all t_i's are positive. We have to show that $A_1 = \ldots = A_n$.

Set $t = t_1$, $A = A_1$, $B = \mathrm{cl}\,(\sum_{i=2}^{n} t_i (1 - t_1)^{-1} A_i)$ and $C = (\sum_{i=1}^{n} t_i A_i^o)^o$. Then

$$tA + (1 - t)B \subset \mathrm{cl}(\sum_{i=1}^{n} t_i A_i) \subset C,$$

because $(\sum_{i=1}^{n} t_i A_i)^o = (\mathrm{cl}(\sum_{i=1}^{n} t_i A_i^o))^{oo} = C^o$. As $B^o \subset (\sum_{i=2}^{n} t_i (1 - t_1)^{-1} A_i^o)_*$ (by Theorem 1), we have that $tA^o + (1 - t)B^o \subset t_1 A_1^o + (\sum_{i=2}^{n} t_i A_i^o)_* \subset (\sum_{i=1}^{n} t_i A_i^o)_* = \mathrm{cl}(\sum_{i=1}^{n} t_i A_i^o) = (\mathrm{cl}(\sum_{i=1}^{n} t_i A_i^o)) = C^o$ (we have used that $M_* + N_* \subset (M + N)_*$ which is true for any two subsets M, N of a linear space). Hence we conclude that $A_1 = A = B = C$, by lemma 4. By the same reasons one sees that also $A_2 = \ldots = A_n = C$.

The proof is completed.

Remark. We hope that our Theorem will find applications in convex analysis.

An easy application is as follows. Let the hypotheses of

Theorem 1 be satisfied and let p be a K-subadditive functional on X' ($K \ge 0$; $p(u + v) \le Kp(u) + Kp(v)$ for all u,v in X'). Then $\sup p((\sum_{i=1}^{n} t_i A_i)^o) \le c(K,n)(\sum_{i=1}^{n} \sup p(t_i A_i^o))$, $\sup p((\sum_{i=1}^{n} t_i A_i)^o) \le K^m(\sum_{i=1}^{n} \sup p(t_i A_i^o) + (2^m - n)p(0))$, where $c(K,n) = \frac{K(2K^{n-1} - K^{n-2} - 1)}{K - 1}$ ($c(K,n) = 1$ if $K = 1$) and m is the first integer such that $n \le 2^m$, provided p is continuous on straight lines.

References

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