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PRODUCTIVE REPRESENTATIONS OF SEMIGROUPS BY PAIRS OF STRUCTURES

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Abstract: We prove that for any commutative semigroup $(S,+)$ there exists a collection $\{r(s) \mid s \in S\}$ of complete metric spaces such that for every $s_1, s_2 \in S$,

(i) $r(s_1 + s_2)$ is isometric to $r(s_1) \times r(s_2)$ and

(ii) if $s_1 \neq s_2$ then $r(s_1)$ is not homeomorphic to $r(s_2)$.

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1. Let us begin with a definition.

Definition. Let $\mathbb{K}$, $\mathbb{H}$ be categories, $\mathbb{K}$ have finite products. Let $\mathcal{F} : \mathbb{K} \longrightarrow \mathbb{H}$ be a functor. Let $(S,+)$ be a commutative semigroup. Any mapping

$$r: S \longrightarrow \operatorname{obj} \mathbb{K}$$

is called an $\mathcal{F}$-productive representation of $(S,+)$ if

(i) for any $s_1, s_2 \in S$, $r(s_1 + s_2)$ is isomorphic to $r(s_1) \times r(s_2)$ in $\mathbb{K}$;

(ii) if $s_1, s_2 \in S$, $s_1 \neq s_2$, then $\mathcal{F}(r(s_1))$ is not isomorphic to $\mathcal{F}(r(s_2))$ in $\mathbb{H}$.

In $[T_2]$, a representation of $(S,+)$ by products in a category

$\mathbb{K}$ is introduced. It is a special case of the above definition with $\mathbb{K} = \mathbb{H}$ and $\mathcal{F}$ = ident. The dual definitions of $\mathcal{F}$-coproductive representation is evident.

2. Some of the known results give $\mathcal{F}$-productive representations of some semigroups. Let us recall some of them.

A) Let $\mathbb{L}$ be the category of lattices and all lattice-homomorphisms; let $\mathbb{L}\,\mathbb{L}$ be the category of all linear lattices and all linear lattice-homomorphisms. Let $\mathcal{L} : \mathbb{L}\,\mathbb{L} \longrightarrow \mathbb{L}$ be the functor which assigns to each linear lattice its underlying lattice. Then

any Abelian group and any countable commutative semigroup have $\mathcal{L}$-productive representations.

B) Let $\mathbb{R}$ be the category of all commutative rings with unit (and all their unit-preserving homomorphisms), let $\mathbb{S}$ be the category of all commutative semigroups with unit. Let $\mathcal{R} : \mathbb{R} \longrightarrow \mathbb{S}$ be the functor which assigns to each ring its multiplicative semigroup. Then

any Abelian group and any countable commutative semigroup have $\mathcal{R}$-productive representations.

C) Let $\mathbb{B}$ be the category of all Banach spaces and all bounded linear operators with the norm ≤ 1, let $\mathbb{B}\mathbb{A}$ be the category of all Banach algebras. Let $\mathcal{B} : \mathbb{B}\mathbb{A} \longrightarrow \mathbb{B}$ be the functor which assigns to each Banach algebra its underlying Banach space. Then

any Abelian group and any countable commutative semigroup have $\mathcal{B}$-productive representations.

In all these cases, the $\mathcal{L}$- or $\mathcal{R}$- or $\mathcal{B}$-productive representations are obtained as follows. By [AKT], aby Abelian group has a representation by coproducts of Boolean spaces (i.e. compact Hausdorff zero-dimensional spaces), in other words, for any Abelian group G there exists a collection $\{r(g) \mid g \in G\}$ of pairwise non-homeomorphic Boolean spaces such that $r(g_1 + g_2)$ is always homeomorphic to the coproduct $r(g_1) \amalg r(g_2)$ of $r(g_1)$ and $r(g_2)$. The analogous result for all countable commutative semigroups is proved in [K] (here, $r(g)$ are metrizable).

Consider the sets $C(r(g))$ of all real-valued continuous functions on these spaces $r(g)$. They can be structured in a lot of ways: As linear lattices and lattices for A), as rings and semigroups for B), as Banach algebras and Banach spaces for C). Structured as a linear lattice or ring or Banach algebra, $C(r(g_1) \amalg r(g_2))$ is isomorphic to $C(r(g_1)) \times C(r(g_2))$ in the corresponding category. Since $r(g_1)$ is not homeomorphic to $r(g_2)$, $C(r(g_1))$ is not isomorphic to $C(r(g_2))$, structured as lattices (by the Birkhoff-Kaplansky theorem) or Banach spaces (by the Banach-Stone theorem) or multiplicative semigroups (by Milgram [M]).

Let us notice that if $\mathcal{F}: \mathbb{K} \longrightarrow \mathbb{H}$ preserves finite products and a semigroup has an $\mathcal{F}$-productive representation, then it has a representation by products in $\mathbb{H}$ in the sense of $[T_2]$. Hence, if a functor $\mathcal{F}$ from an arbitrary category into the category Set of all sets or into the category Lin of all linear spaces preserves finite products, then no non-trivial Abelian group has an $\mathcal{F}$-productive representation.

3. Let $\mathbb{CM}$ be the category of all complete metric spaces with diameter ≤ 1 and all their contractions (we re-

call that a mapping c is a contraction if dist $(c(x),c(y)) \leq$ $\leq$ dist (x,y) for all x,y. Let us notice that isomorphisms in $\mathbb{C}\mathbb{M}$ coincide with isometries and a product-metric d of d_1 and d_2 is given by the usual formula

$$d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\} .$$

Let Top be the category of all topological spaces and all their continuous mappings. Here isomorphisms coincide with homeomorphisms. Let

$$\mathcal{M} : \mathbb{C}\mathbb{M} \longrightarrow \mathbb{T}\mathrm{op}$$

be the functor which assigns to each metric space its underlying topological space. The aim of this note is to prove the following theorem:

<u>Theorem</u>. Every commutative semigroup has an $\mathcal{M}$-productive representation.

Every commutative semigroup has a representation by products of uniform, proximity and topological spaces; by [AK], every C-embeddable semigroup has a representation by products of metrizable topological spaces. The above theorem strengthens all these results.

4. First, we sketch modifications of the general method, described in $[T_2]$. If a semigroup S has an $\mathcal{F}$-productive representation, then any of its subsemigroups has also an $\mathcal{F}$-productive representation. Consequently, it is sufficient to investigate $\mathcal{F}$-productive representation of "universal semigroups" (this means universal for some class of semigroups with respect to an embedding of semigroups).
Denote by N the additive semigroup of all non-negative inte-

gers, by N^{m} its m-th power (with the operation given pointwise) and by $\exp N^{m}$ the semigroup of all its subsets (with the operation given by $A + B = \{a + b \mid a \in A, b \in B\}$). By [$T_3$], any commutative semigroup S can be embedded in $\exp N^{m}$ with $m = \aleph_0 \cdot \operatorname{card} S$. Hence, we shall investigate $\mathcal{F}$-productive representations of the semigroups $\exp N^{m}$.

5. We shall use the following notation and conventions. Isomorphism in a category will be denoted by $\simeq$, product by $\prod$ (or $\times$ for finite collections), coproduct by $\amalg$. The product of the empty collection is a terminal object (it can be added to a category whenever it is missing).
If a is an arbitrary object of a category with finite products, then a^0 is the terminal object, $a^1 \simeq a$, $a^{n+1} \simeq a \times a^n$.
We say that a category $\mathbb{K}$ with all products and coproducts is <u>distributive</u> (see [T_2]) if

$$(\coprod_{i \in I} a_i) \times (\coprod_{j \in J} b_j) \simeq \coprod_{(i,j) \in I \times J} (a_i \times b_j).$$

We say that an object a is a <u>summand</u> of b if $b \simeq a \amalg c$ for an object c.

6. Let $\mathbb{K}$ be a category with products, let $\mathcal{F} : \mathbb{K} \longrightarrow \mathbb{H}$ be a functor. Let $\mathfrak{Z}$ be a set of objects of $\mathbb{K}$. For any $f \in N^{\mathfrak{Z}}$, denote by $\mathbf{Z}(f)$ the product $\prod_{Z \in \mathfrak{Z}} Z^{f(Z)}$ (if $f(Z) = 0$, $Z^{f(Z)}$ is the terminal object). We say that $\mathfrak{Z}$ is an <u>$\mathcal{F}$-independent set of objects of</u> $\mathbb{K}$ if for every $f \in N^{\mathfrak{Z}}$, $A \subset N^{\mathfrak{Z}}$,

$f \in A$ whenever $\mathcal{F}(\mathbb{Z}(f))$ is a summand of $\mathcal{F}(\coprod_{(g,s)\in A\times S} \mathbb{Z}(g)_s)$, where S is a set and $\mathbb{Z}(g)_s \simeq \mathbb{Z}(g)$ for all $s \in S$.

(This generalizes the notion of productively independent set of objects, see $[T_2]$ and [AK].)

7. Proposition. Let $\mathbb{K}$ be a distributive category, let $\mathcal{F}: \mathbb{K} \longrightarrow \mathbb{H}$ preserve coproducts. Let there exist an $\mathcal{F}$-independent set $\mathcal{Z}$ of objects of $\mathbb{K}$. Then the semigroup $\exp N^{\mathcal{Z}}$ has an $\mathcal{F}$-productive representation.

Proof. For any $f \in N^{\mathcal{Z}}$ denote $\mathbb{Z}(f) = \prod_{Z\in\mathcal{Z}} Z^{f(Z)}$; let $\mathbb{C}(f)$ be a coproduct of 2^{m} copies of $\mathbb{Z}(f)$ with $m = \operatorname{card} \mathcal{Z}$. For $A \subset N^{\mathcal{Z}}$ put

$$r(A) = \coprod_{f\in A} \mathbb{C}(f).$$

Then r is an $\mathcal{F}$-productive representation of $\exp N^{\mathcal{Z}}$. For, if $A, B \subset N^{\mathcal{Z}}$, then $r(A + B) \simeq r(A) \times r(B)$ (implied by $\mathbb{Z}(f + g) \simeq \mathbb{Z}(f) \times \mathbb{Z}(g)$). If $A \neq B$, say if $A \setminus B \neq \emptyset$, then, for $f \in A \setminus B$, $\mathcal{F}\mathbb{Z}(f)$ is a summand of $\mathcal{F}r(A)$ while it cannot be a summand of $\mathcal{F}r(B)$ because $\mathcal{Z}$ is $\mathcal{F}$-independent. Hence, $\mathcal{F}r(A)$ is not isomorphic to $\mathcal{F}r(B)$ in $\mathbb{H}$.

Corollary. Let $\mathbb{K}$ be a distributive category, let $\mathcal{F}: \mathbb{K} \longrightarrow \mathbb{H}$ be a coproduct-preserving functor. Let $\mathbb{K}$ have an arbitrarily large $\mathcal{F}$-independent set of objects. Then any commutative semigroup has an $\mathcal{F}$-productive representation.

8. Let us examine the category $\mathbb{CM}$. It has all coproducts (for, if $\{(X_i, d_i) \mid i \in I\}$ is a collection of ob-

jects, X_i disjoint, put $X = \bigcup_{i \in I} X_i$, $d(x,y) = d_i(x,y)$ whenever $x,y \in X_i$ for some i, $d(x,y) = 1$ otherwise; d is complete whenever all the d_i's are complete; (X,d) is a coproduct in $\mathbb{CM}$). It has all products (for, if $\{(X_i,d_i) \mid i \in I\}$ is a collection of objects, put $X = \prod_{i \in I} X_i$, $d(\{x_i\}\{y_i\}) = \sup_{i \in I} d_i(x_i,y_i)$). Clearly, $\mathbb{CM}$ is distributive. The functor $\mathcal{M} : \mathbb{CM} \longrightarrow \mathbf{Top}$ preserves coproducts and finite products, but it does not preserve products in general. To prove the theorem, we have to show that $\mathbb{CM}$ contains arbitrarily large sets of $\mathcal{M}$-independent sets of objects.

9. If $\{Y_i \mid i \in I\}$ is a collection of topological spaces, denote by $\mathop{\mathcal{B}}_{i \in I} Y_i$ their box-product. We recall that a set $\mathcal{Y}$ of topological spaces is called <u>stiff</u> if for any Y_1, $Y_2 \in \mathcal{Y}$ and any continuous mapping $m: Y_1 \longrightarrow Y_2$ either m is constant or $Y_1 = Y_2$ and $m = \mathrm{ident.}$

Now, let $\mathcal{Y}$ be a set of topological spaces. For any $f \in N^{\mathcal{Y}}$ denote by $\mathbb{B}(f)$ a topological space with the same underlying set as $\mathop{\mathcal{B}}_{Y \in \mathcal{Y}} Y^{f(Y)}$ and such that both the identical mappings

$$\mathop{\mathcal{B}}_{Y \in \mathcal{Y}} Y^{f(Y)} \longrightarrow \mathbb{B}(f) \longrightarrow \prod_{Y \in \mathcal{Y}} Y^{f(Y)}$$

are continuous (where $\prod$ denotes product in $\mathbf{Top}$).

In the following proposition, S is a set and, for each $s \in S$, $(\mathbb{B}(g))_s$ is homeomorphic to $\mathbb{B}(g)$. If $X \subset \mathbb{B}(g)$, X_s means the corresponding subspace of $(\mathbb{B}(g))_s$.

<u>Proposition</u>. Let $\mathcal{Y}$ be a stiff set of connected Hausdorff spaces. Let $f \in N^{\mathcal{Y}}$, $A \subset N^{\mathcal{Y}}$ be given. If $\mathbb{B}(f)$ is homeomorphic to a closed-and-open subset of $\coprod_{(g,s) \in A \times S} (\mathbb{B}(g))_s$,

then $f \in A$.

Proof. a) First, let us notice that for any $Y \in \mathcal{Y}$ and any $m_1, m_2 \in N$, the existence of a homeomorphism of Y^{m_1} into Y^{m_2} implies $m_1 \leq m_2$ (see [H]).

b) For any $g \in N^{\mathcal{Y}}$ and any $x \in \mathbb{B}(g)$, denote by $\mathbb{B}_x(g)$ the subspace of $\mathbb{B}(g)$ consisting of all these points y which differ from x only in finitely many coordinates. Clearly, every $\mathbb{B}_x(g)$ is connected. One can see easily by a), that if, for some $g_1, g_2 \in N^{\mathcal{Y}}$ and some $x \in \mathbb{B}(g_1)$, there exists a homeomorphism of $\mathbb{B}_x(g_1)$ into $\mathbb{B}(g_2)$, then $g_1 \leq g_2$.

c) Now, let $h: \mathbb{B}(f) \longrightarrow \coprod_{(g,s) \in A \times S} (\mathbb{B}(g))_s$ be a homeomorphism onto a closed-and-open subset. Choose $x \in \mathbb{B}(f)$. Since $\mathbb{B}_x(f)$ is connected, there exists $(g_0, s_0) \in A \times S$ such that $h(\mathbb{B}_x(f)) \subset (\mathbb{B}(g_0))_{s_0}$. By b), $f \leq g_0$. Put $y = h(x)$. Since $h(\mathbb{B}(f))$ is a closed-and-open set containing y and $(\mathbb{B}_y(g_0))_{s_0}$ is connected, it is contained in $h(\mathbb{B}(f))$. Hence, h^{-1} defines a homeomorphism of $\mathbb{B}_y(g_0)$ into $\mathbb{B}(f)$. By b), $g_0 \leq f$. We conclude that $f = g_0 \in A$.

10. By [T_1], there exist arbitrarily large stiff sets $\mathcal{Y}$ of connected topological spaces such that any $Y \in \mathcal{Y}$ can be metrized by a complete metric, say d_Y. We may suppose $d_Y \leq 1$. By the previous proposition, $\mathcal{X} = \{(Y, d_Y) \mid Y \in \mathcal{Y}\}$ is an $\mathcal{M}$-independent set of objects of $\mathbf{CM}$. This completes the proof of the theorem.

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