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COMPLETION OF SEQUENTIAL CAUCHY SPACES

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Abstract: We study two types of sequential Cauchy spaces projectively generated by classes of functions, their completions, and their mutual relations.

Key words: Sequential Cauchy space, completion, convergence space, sequential envelope.

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1. Introduction. For the reader's convenience we recall in this section some basics about (sequential) Cauchy spaces.

Notation 1.1. If $\langle x_n \rangle$, $\langle y_n \rangle$ are two sequences, then $\langle x_n \rangle \wedge \langle y_n \rangle$ denotes a sequence $\langle z_n \rangle$ defined as follows: $z_1 = x_1$, $z_2 = y_1$, $z_3 = x_2$, $z_4 = y_2, \ldots$, i.e. $x_n = z_{2n-1}$, $y_n = z_{2n}$.

Definition 1.2. A Cauchy space is a pair (X, L), where X is a set and L a collection of sequences ranging in X such that

(1) $\langle x \rangle \in L$ for each $x \in L$;

(2) $\langle x_n \rangle \in L$ implies $\langle x'_n \rangle \in L$ for each subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$;

(3) if $\langle x_n \rangle, \langle y_n \rangle \in L$ and there are subsequences $\langle x'_n \rangle$ of $\langle x_n \rangle$ and $\langle y'_n \rangle$ of $\langle y_n \rangle$ such that $x'_n = y'_n$, $n \in N$, then $\langle x_n \rangle \wedge \langle y_n \rangle \in L$; and

(4) if $\langle x_n \rangle \wedge \langle x \rangle \in L$ and $\langle x_n \rangle \wedge \langle y \rangle \in L$, then $x = y$.
If (X,L) is a Cauchy space, then L is called a Cauchy structure for X. If L satisfies the additional condition

(5) $\langle x_n \rangle \in L$ whenever

(a) each subsequence $\langle x_n' \rangle$ of $\langle x_n \rangle$ contains a subsequence $\langle x_n'' \rangle$ of $\langle x_n' \rangle$ such that $\langle x_n'' \rangle \in L$; and

(b) if $\langle x_n' \rangle$ and $\langle x_n'' \rangle$ are subsequences of $\langle x_n \rangle$ such that $\langle x_n' \rangle$, $\langle x_n'' \rangle \in L$, then $\langle x_n' \rangle \wedge \langle x_n'' \rangle \in L$;

then (X,L) is said to be a $*$ Cauchy space.

The effect of condition (5) can be brought out by considering the real line with its usual metric. Every bounded sequence of real numbers has a Cauchy subsequence. Hence, every bounded sequence of real numbers satisfies condition (a). Yet every bounded sequence of real numbers is not Cauchy in the usual sense because (b) is lacking; e.g. consider the sequence 0, 1, 0, 1, 0, 1, ...

A Cauchy space (X,L) induces a convergence space $(X,\mathcal{L},\lambda)$ in the following natural way: $x = \mathcal{L}\text{-}\lim x_n$ iff $\langle x_n \rangle \wedge \langle x \rangle \in L$. Moreover, if (X,L) is a $*$ Cauchy space, then $\mathcal{L} = \mathcal{L}^*$. The topological modification λ^{ω_1} of λ will be called a topological closure for X. A subspace Y of X is topologically dense in X if $\lambda^{\omega_1} Y = X$. A Cauchy space is said to be complete if each Cauchy sequence converges in the induced convergence space. A mapping $f\colon (X_1,L_1) \longrightarrow (X_2,L_2)$ is said to be Cauchy-continuous if $\langle x_n \rangle \in L_1$ implies $\langle f(x_n) \rangle \in L_2$. The set of all Cauchy-continuous functions on (X,L) is denoted by $\hat{C}(X,L)$. The set

$M = \{\langle f_m \rangle \in (\hat{C}(X,L))^N;\ \lim_{n,m\to\infty} f_m(x_n)$ exists for each $\langle x_n \rangle \in L\}$ is a Cauchy structure for $\hat{C}(X,L)$ and is said to be the continuous Cauchy structure. The space $(\hat{C}(\hat{C}(X,L),M),M)$ is denoted by $(\hat{C}^2(X,L),M^2)$. The evaluation mapping $ev_X: (X,L) \longrightarrow (\hat{C}^2(X,L),M^2)$ is defined by $ev_X(x) = \Phi_x$, where for $f \in \hat{C}(X,L)$ we define $\Phi_x(f) = f(x)$; it is always Cauchy-continuous. If it is a Cauchy-embedding (i.e. a Cauchy-homeomorphism into), then (X,L) is said to be $\hat{C}$-embedded.

2. Projective generations of Cauchy structures.

Proposition and definition 2.1. Let (X,L) be a Cauchy space and $D \subset \hat{C}(X,L)$, D separates points of X. Let $L_D = \{\langle x_n \rangle \in X^N;\ \lim f(x_n)$ exists whenever $f \in D\}$ and $L_d = \{\langle x_n \rangle \in X^N;\ \lim_{m,n\to\infty} f_m(x_n)$ exists whenever $\langle f_m \rangle$, $f_m \in D$ is a Cauchy sequence in $(\hat{C}(X,L),M)\}$. Then L_D and L_d are Cauchy structures for X and $L \subset L_d \subset L_D$. If $L = L_D$, then L, resp. (X,L), is said to be projectively generated by D. If $L = L_d$, then L, resp. (X,L), is said to be c-projectively generated by D.

It follows immediately that if a space is projectively generated by D, then it is also c-projectively generated by D. The converse statement is not true in general as it will be shown by a counterexample (see Proposition 4.7). In [I - K] it was proved that for $D = C(X,L)$ the following are equivalent:
(a) (X,L) is $\hat{C}$-embedded; (b) $L = L_D$; (c) $L = L_d$ (the original notation is $L_D = L_{\hat{F}}$, $L_d = L_M$).

3. d-completion.

Definition 3.1. Let (X,L) be a Cauchy space c-projectively generated by $D \subset \hat{C}(X,L)$. A complete Cauchy space (X_1,L_1) is said to be a d-completion of (X,L) if

(a) (X,L) is a topologically dense subspace of (X_1,L_1);

(b) for each $f \in D$ there is $\bar{f} \in \hat{C}(X_1,L_1)$ such that $f = \bar{f} \mid X$, i.e. $D \subset \hat{C}(X_1,L_1) \mid X$;

(c) (X_1,L_1) is c-projectively generated by $\bar{D} = \{\bar{f} \in \hat{C}(X_1,L_1);\ \bar{f} \mid X \in D\}$; and

(d) $\bar{D}$ and D endowed with the corresponding continuous Cauchy structures are Cauchy-homeomorphic under the natural correspondence, i.e. the correspondence $\bar{f} \longrightarrow \bar{f} \mid X = f$ is one-to-one and $\langle \bar{f}_n \rangle$, $\bar{f}_n \in \bar{D}$, is a Cauchy sequence in $(\hat{C}(X_1,L_1),M)$ iff $\langle f_n \rangle$, $f_n = \bar{f}_n \mid X$, is a Cauchy sequence in $(\hat{C}(X,L),M)$.

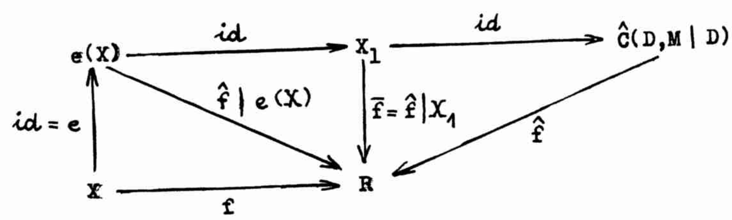
Lemma 3.2. Let (X,L) be a Cauchy space c-projectively generated by $D \subset \hat{C}(X,L)$, $(D, M \mid D)$ the subspace of $(\hat{C}(X,L),M)$, and e a mapping of (X,L) into $(\hat{C}(D,M \mid D),M)$ defined as follows: $e(x) = \Phi_x$, where for $f \in D$ we define $\Phi_x(f) = f(x)$. Then e is a Cauchy embedding.

Lemma 3.2 was proved in [I - K] in the special case of $D = \hat{C}(X,L)$. The proof of the general case is similar.

Theorem 3.3. Let (X,L) be a Cauchy space c-projectively generated by $D \subset \hat{C}(X,L)$. Then there exists a d-completion of (X,L).

Proof. It follows from Lemma 3.2 that identifying x with $e(x)$ we can consider (X,L) as a subspace of $(\hat{C}(D,M \mid D),M)$. We shall prove that the subspace (X_1,L_1) of $(\hat{C}(D,M \mid D),M)$, where

X_1 is the topological closure of X in $(\hat{C}(D,M \mid D),M)$ and $L_1 = M \mid X_1$, is a d-completion of (X,L). It was proved in [I - K] that $(\hat{C}(D,M \mid D),M)$ is a complete space. Thus the closed subspace (X_1,L_1) of $(\hat{C}(D,M \mid D),M)$ is complete. We are to prove that (X_1,L_1) satisfies conditions (a) - (d) of Definition 3.1. Condition (a) follows from the construction of (X_1,L_1). It was proved in [F] that the space $(\hat{C}(X,L),M)$ is $\hat{C}$-embedded. Thus the subspace $(D,M \mid D)$ is also $\hat{C}$-embedded, and hence the evaluation mapping ev_D: $(D,M \mid D) \longrightarrow (\hat{C}^2(D,M \mid D),M^2)$ is a Cauchy embedding. Consequently, for each $f \in D$ the image $ev_D(f) = \hat{f}$ is a Cauchy-continuous function on $(\hat{C}(D,M \mid D),M)$. Since $\hat{f}(\Phi) = \Phi(f)$ for each $\Phi \in \hat{C}(D,M \mid D)$, we have $\hat{f}(x) = f(x)$ for each $\Phi_x = x \in X$. Hence $\bar{f} = \hat{f} \mid X_1$ is a Cauchy-continuous extension of f onto (X_1,L_1) and condition (b) is satisfied. The construction of $\bar{f}$ is shown on the following diagram:



Now, we shall prove condition (d). It follows by a standard topological argument that the extension $\bar{f}$ of f is uniquely determined. Hence the natural correspondence $\bar{f} \longrightarrow \bar{f} \mid X = f$ is one-to-one. Clearly, if $\langle \bar{f}_n \rangle$, $\bar{f}_n \mid X \in D$, is a Cauchy sequence in $(\hat{C}(X_1,L_1),M)$, then $\langle f_n \rangle$, $f_n = \bar{f}_n \mid X$, is a Cauchy sequence in $(\hat{C}(X,L),M)$. Conversely, let $\langle f_n \rangle$ be a Cauchy sequence in $(D,M \mid D)$. Since ev_D is a Cauchy embedding, the sequence $\langle \hat{f}_n \rangle$, $\hat{f}_n = ev_D(f_n)$, is a Cauchy sequence in $(\hat{C}^2(D,M \mid D),M^2)$. Hence $\langle \bar{f}_n \rangle$, $\bar{f}_n \mid X = f_n$, is a Cauchy sequence in $(\hat{C}(X_1,L_1),M)$.

It remains to prove condition (c). Let $\langle \Phi_n \rangle$ be a sequence in $X_1 \subset \hat{C}(D,M \mid D)$ such that

(1) $\lim_{m,n\to\infty} \bar{f}_m(\Phi_n)$ exists whenever $\langle \bar{f}_m \rangle$,$\bar{f}_m \in \bar{D}$,is a Cauchy sequence in $(\hat{C}(X_1,L_1),M)$.

Since $\bar{f}_m(\Phi_n) = \Phi_n(f_m)$, $f_m = \bar{f}_m \mid X$, it follows from (d) that (1) is equivalent to

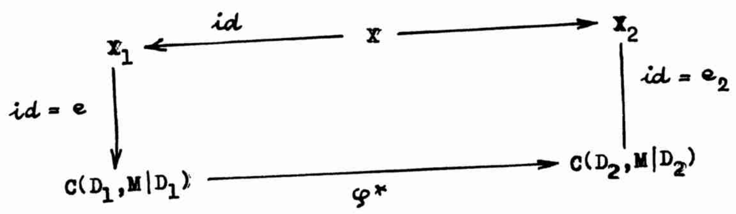
(2) $\lim_{m,n\to\infty} \Phi_n(f_m)$ exists whenever $\langle f_m \rangle$ is a Cauchy sequence in $(D,M \mid D)$.

Thus $\langle \Phi_n \rangle \in L_1$ and the proof is complete.

<u>Theorem 3.4</u>. Let (X,L) be a Cauchy space c-projectively generated by $D \subset \hat{C}(X,L)$. If (X_1,L_1) and (X_2,L_2) are two d-completions of (X,L), then there is a Cauchy homeomorphism $h: (X_1,L_1) \longrightarrow (X_2,L_2)$ such that $h(x) = x$ for each $x \in X$.

<u>Proof</u>. For $i = 1,2$, denote by $D_i = \{ f \in \hat{C}(X_i,L_i);\ f \mid X \in D \}$, by $(D_i,M \mid D_i)$ the subspace of $(\hat{C}(X_i,L_i),M)$, and by $(D,M \mid D)$ the subspace of $(\hat{C}(X,L),M)$. It follows from (d) in Definition 3.2 that $(D_i,M \mid D_i)$ and $(D,M \mid D)$ are Cauchy-homeomorphic under the natural correspondence. Consequently, $\varphi : (D_2,M \mid D_2) \longrightarrow (D_1,M \mid D_1)$, where for $f \in D_2$ its image $\varphi(f)$ is determined by $\varphi(f) \mid X = f \mid X$, and hence also its first conjugate $\varphi^* : (\hat{C}(D_1,M \mid D_1),M) \longrightarrow (\hat{C}(D_2,M \mid D_2),M)$, $\varphi^*(\Phi) = \varphi \circ \Phi$, are Cauchy homeomorphisms. It follows from Lemma 3.2 that identifying x with $e_i(x)$, where for $f \in D_i$ we define $(e_i(x))(f) = f(x)$, we can consider the complete space (X_i,L_i) as a closed subspace of $(\hat{C}(D_i,M \mid D_i),M)$.

Now, an easy computation shows that for each $x \in X$ we have $\varphi^*(x) = x$.



Since X is topologically dense in (X_1, L_1), it follows by a standard topological argument that $h = \varphi^* | X_1$ is the desired Cauchy homeomorphism.

4. D-completion.

<u>Definition 4.1</u>. Let (X,L) be a Cauchy space projectively generated by $D \subset \hat{C}(X,L)$. A complete Cauchy space (X_1, L_1) is said to be a D-completion of (X,L) if

(a) (X,L) is a topologically dense subspace of (X_1, L_1);

(b) for each $f \in D$ there is $\bar{f} \in \hat{C}(X_1, L_1)$ such that $f = \bar{f} | X$, i.e. $D \subset \hat{C}(X_1, L_1) | X$; and

(c) (X_1, L_1) is projectively generated by $\bar{D} = \{\bar{f} \in \hat{C}(X_1, L_1);\ \bar{f} | X \in D\}$.

<u>Proposition 4.2</u>. Let (X,L) be a Cauchy space projectively generated by $D \subset \hat{C}(X,L)$ and $(X, \mathcal{L}^*, \lambda)$ the associated convergence space. **Then:**

(a) $D \subset C(X)$ and $(X, \mathcal{L}^*, \lambda)$ is D-sequentially regular.

(b) L is the set of all D-fundamental sequences in $(X, \mathcal{L}^*, \lambda)$.

(c) $(X, \mathcal{L}^*, \lambda)$ is D-sequentially complete iff (X,L)

is complete.

The straightforward proof is omitted.

<u>Proposition 4.3</u>. Let $(X,\mathcal{L}^*,\lambda)$ be a D-sequentially regular convergence space and L the set of all D-fundamental sequences. Then:

(a) L is a $*$Cauchy structure for X.

(b) $D\subset \hat{C}(X,L)$ and (X,L) is projectively generated by D.

(c) $(X,\mathcal{L}^*,\lambda)$ is associated with (X,L).

(d) (X,L) is complete iff $(X,\mathcal{L}^*,\lambda)$ is D-sequentially complete.

The straightforward proof is omitted.

<u>Theorem 4.4</u>. Let (X,L) be a Cauchy space projectively generated by $D\subset \hat{C}(X,L)$. Then there exists a D-completion of (X,L).

<u>Proof</u>. Let $(X,\mathcal{L},\lambda)$ be the convergence space associated with (X,L). It follows from (a) in Proposition 4.2 that $(X,\mathcal{L},\lambda)$ is D-sequentially regular. Let $(X_1,\mathcal{L}_1,\lambda_1)$ be a D-sequential envelope of $(X,\mathcal{L},\lambda)$, $\bar{D}=\{\bar{f}\in C(X_1);\ \bar{f}\,|\,X\in D\}$, and L_1 the set of all $\bar{D}$-fundamental sequences in X_1. It follows from Proposition 4.2 and Proposition 4.3 that (X_1,L_1) is a D-completion of (X,L).

<u>Note 4.5</u>. Let $(X,\mathcal{L}^*,\lambda)$ be a D-sequentially regular convergence space. Let L be the set of all D-fundamental sequences in X. It follows from Proposition 4.3 that (X,L) is a $*$Cauchy space projectively generated by $D\subset \hat{C}(X,L)$. Let (X_1,L_1) be a D-completion of (X,L). Using Proposition 4.2 and Proposition 4.3 it is easy to see that the convergence space $(X_1,\mathcal{L}_1,\lambda_1)$ associated with (X_1,L_1) is a D-sequential enve-

lope of $(X,\mathcal{L}^*,\lambda)$.

<u>Theorem 4.6</u>. Let (X,L) be a Cauchy space projectively generated by $D\subset \hat{C}(X,L)$. If (X_1,L_1) and (X_2,L_2) are two D-completions of (X,L), then there is a Cauchy homeomorphism $h: (X_1,L_1)\longrightarrow(X_2,L_2)$ such that for each $x\in X$ we have $h(x) = x$.

<u>Proof</u>. Let $(X,\mathcal{L},\lambda)$ be the convergence space associated with (X,L) and $(X_i,\mathcal{L}_i,\lambda_i)$ the convergence space associated with (X_i,L_i), $i = 1,2$. It follows from Note 4.5 that $(X_i,\mathcal{L}_i,\lambda_i)$ is a D-sequential envelope of $(X,\mathcal{L},\lambda)$. Hence there is a homeomorphism $h: (X_1,\mathcal{L}_1,\lambda_1)\longrightarrow(X_2,\mathcal{L}_2,\lambda_2)$ such that for each $x\in X$ we have $h(x) = x$ (cf. Theorem 5 in [N]). Since (X_i,L_i) are complete space, $h: (X_1,L_1)\longrightarrow(X_2,L_2)$ is a Cauchy homeomorphism.

5. <u>Example</u>.

<u>Definition 5.1</u>. Let $X\neq\emptyset$ and $\langle x_n\rangle,\langle y_n\rangle\in X^N$. We say that $\langle y_n\rangle$ <u>is derived from</u> $\langle x_n\rangle$, in symbols $\langle y_n\rangle \dashv \langle x_n\rangle$, if $F(\langle y_n\rangle)\supset F(\langle x_n\rangle)$, where $F(\langle z_n\rangle)$ denotes the filter generated by sections of a sequence $\langle z_n\rangle$.

<u>Example 5.2</u>. Let $X_2 = (\bigcup_{m\in N}\bigcup_{n\in N}(x_{mn}))\cup(\bigcup_{m\in N}(x_m))\cup \cup(x_0)$. Let $\alpha\in N^N$, $m_0\in N$, $A\subset(\bigcup_{m\in N}(x_{m\alpha(m)}))$, and $f\in\{0,1\}^{X_2}$ a function on X_2 defined as follows:

$$f(x) = 1 \text{ for } x\in(A\cup(\bigcup_{n\in N}(x_{m_0 n}))\cup(x_{m_0})),$$

$$f(x) = 0 \text{ otherwise.}$$

Let $\bar{D}$ be the set of all such functions and (X_2,L_2) the Cauchy space projectively generated by $\bar{D}$. Let $X = \bigcup_{m\in N}\bigcup_{n\in N}(x_{mn})$,

$X_1 = X \cup (\bigcup_{m \in N} (x_m))$, $L = L_2 \mid X$, $L_1 = \{\langle z_n \rangle \in X_1^N;\ \langle z_n \rangle \prec \langle x \rangle$, $x \in X_1$, or $\langle z_n \rangle \prec (\langle x_{mn} \rangle \wedge \langle x_m \rangle)$, $m \in N\}$, and $D = \overline{D} \mid X$.

Since (X,L) is clearly projectively generated by D it is also c-projectively generated by D and hence (X,L) possesses both a D-completion and a d-completion.

<u>Proposition 5.3</u>. For $\hat{D} = \overline{D} \mid X_1$ the space (X_1, L_1) is c-projectively generated by $\hat{D}$, but not projectively generated by $\hat{D}$.

<u>Hint</u>. $L_2 = \{\langle z_n \rangle \in X_2^N;\ \langle z_n \rangle \prec \langle x \rangle$, $x \in X_2$, or $\langle z_n \rangle \prec (\langle x_{mn} \rangle \wedge \langle x_m \rangle)$, or $\langle z_n \rangle \prec (\langle x_m \rangle \wedge \langle x_o \rangle)\}$ and $\langle x_m \rangle \in (L_2 \mid X_1 - L_1)$.

<u>Proposition 5.4</u>. (X_1, L_1) is a d-completion of (X,L).

<u>Hint</u>. $L = \{\langle z_n \rangle \in X^N;\ \langle z_n \rangle \prec \langle x \rangle$, $x \in X$, or $\langle z_n \rangle \prec \langle x_{mn} \rangle$, $m \in N\}$.

<u>Proposition 5.5</u>. (X_2, L_2) is a D-completion of (X,L).

<u>Proposition 5.6</u>. $\overline{D}$ and D endowed with the corresponding continuous Cauchy structures are not Cauchy-homeomorphic under the natural correspondence.

<u>Proof</u>. For otherwise (X_2, L_2) would be also a d-completion of (X,L), which would imply the existence of a Cauchy homeomorphism $h: (X_1, L_1) \longrightarrow (X_2, L_2)$ such that for each $x \in X$ we have $h(x) = x$.

<u>Note 5.7</u>. This shows that the condition (d) in Definition 3.1 is necessary and sufficient for the uniqueness of the d-completion up to a commuting Cauchy homeomorphism (cf. Theorem 3.4).

Note 5.8. Let (X,L) be a $\hat{C}$-embedded Cauchy space. Since for $D = \hat{C}(X,L)$ we have $L = L_d = L_D$, it follows immediately that a d-completion of (X,L) is also a D-completion of (X,L). Consequently, the two completions are equivalent. It might be of some interest to characterize classes $D \subset \hat{C}$ for which the two completions are equivalent.

R e f e r e n c e s

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