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ON EXTENSIONS OF FUNCTORS TO THE KLEISLI CATEGORY

Jiří VINÁREK, Praha

Abstract: Sums of Hom(n,-) with n bounded cannot be extended on a Kleisli category of the monad Mon corresponding to the variety of monoids. On the other hand, the countable sum $\bigvee_{n=1}$ Hom(n,-) can be extended on this Kleisli category.

Key words: Set functor, hom-functor, monad, Kleisli category, distributive laws.

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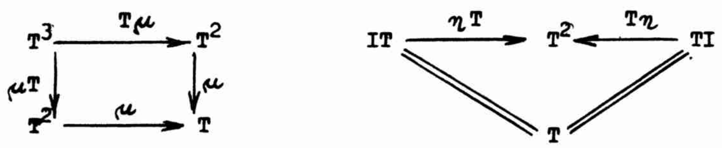
In [1], M.A. Arbib and E.G. Manes studied a problem when a functor $F: \mathcal{K} \longrightarrow \mathcal{K}$ could be extended to the Kleisli category of a monad. They proved that a sufficient and necessary condition for existence of such an extension is commuting of diagrams analogous to the Beck distributive laws between monads (see [2]). Therefore, the term "distributive laws" is used for these diagrams, too.

M.A. Arbib and E.G. Manes proved in [1] that set functors $- \times \Sigma$ satisfy these distributive laws with respect to any monad over the category Set of sets and mappings and therefore they can be extended on a Kleisli category of any monad. In the present note, there is shown that a similar assertion is not true already for some hom-functors and for very natural monads. Such a very naturally defined monad is a monad corresponding to the variety of monoids (i.e. semigroups with units) which does not satisfy distributive laws with respect to Hom(2,-) (more generally, with respect to sums of Hom(n,-) with n bounded - see Proposition 1.1). On the other hand, this monad satisfies distributive laws with respect to the countable sum $\bigvee_{n=1}^{+\infty}$ Hom(n,-) (see Proposition 1.3).

I am indebted to V. Trnková for an impulse to consider problems mentioned and for valuable advice.

0. At first, we recall some definitions and establish notations.

0.1. Let $\mathcal{K}$ be a category, $T: \mathcal{K} \longrightarrow \mathcal{K}$ a functor, $I: \mathcal{K} \longrightarrow \mathcal{K}$ an identity functor, $\eta : I \longrightarrow T$, $\mu : T^2 \longrightarrow T$ natural transformations. We recall that (T, η, μ) is called a monad iff the following diagrams commute:



0.2. <u>Notations.</u> a) Denote Mon = (M,e,m) a monad which assigns to each set A a free monoid over A. (I.e. MA = $= \{a_1 \dots a_n;\ n \in \{1, \dots\},\ a_i \in A \text{ for } i = 1, \dots, n\} \cup \{\Lambda\}$, where Λ is the empty word, $e_A(a) = a$, $m_A((a_{11} \dots a_{1k_1}) \dots$ $\dots (a_{n1} \dots a_{nk_n})) = a_{11} \dots a_{nk_n}$). The corresponding category of monadic algebras is a variety of all the monoids, the corresponding Kleisli category is its subcategory of free monoids.

b) Q_n denotes a functor which assigns to each set A a set A^n of n-tuples of its elements and which is obviously defined on mappings.

c) exp A denotes the set of all the subsets of A.

0.3. We recall the following definition (Arbib-Manes): Let $\mathcal{K}$ be a category, F: $\mathcal{K} \longrightarrow \mathcal{K}$ a functor, (T, η, μ) a monad. F is said to satisfy distributive laws over (T, η, μ) if there exists an assignment to each object A of $\mathcal{K}$ a morphism λ_A: FTA $\longrightarrow$ TFA such that the following two diagrams commute for each A and α : A $\longrightarrow$ TB.

(1)

$$\begin{array}{ccc} FTA & \xrightarrow{\lambda_A} & TFA \\ & {\scriptstyle F(\eta_A)} \nwarrow \quad \nearrow {\scriptstyle \eta_{FA}} & \\ & FA & \end{array}$$

(the first distributive law)

(2)

$$\begin{array}{ccc} FTA & \xrightarrow{\lambda_A} & TFA \\ {\scriptstyle F(\alpha^{\#})} \downarrow & & \downarrow {\scriptstyle (\lambda_B \circ F(\alpha))^{\#}} \\ FTB & \xrightarrow{\lambda_B} & TFB \end{array}$$

(the second distributive law)

where $\alpha^{\#} = \mu_B \circ T(\alpha)$.

0.4. <u>Remark</u>. A functor F can be extended on a Kleisli category over (T, η, μ) iff it satisfies the distributive laws over (T, η, μ).

1.1. Proposition. Let $I \neq \emptyset$ be a set, $\underline{N}$ be a set of all the natural numbers, $\varphi : I \longrightarrow \underline{N}$ be a bounded mapping, $n = \max_{i \in I} \varphi(i) \geq 2$. Then $F = \bigvee_{i \in I} Q_{\varphi(i)}$ does not satisfy distributive laws over Mon.

Proof. Suppose existence of a collection $\{ \lambda_A : FMA \longrightarrow MFA;\ A \in \mathrm{obj}\ \underline{Set} \}$ such that the distributive laws hold.

I. Choose sets $A_0, \ldots, A_n$, A such that $A_0 \subseteq \ldots \subseteq A_n \subseteq A$,

card $A_0 = 1$,

card $A_j > n \cdot \sum_{i \in I} (\mathrm{card}\ A_{j-1} + n - j + 3)^{\varphi(i)}$ for $j = 1, \ldots, n-2$,

card $A_{n-1} > n \cdot \sum_{i \in I} (\mathrm{card}\ A_{n-2} + 3)^{\varphi(i)} + 1$,

card $A_n > n \cdot \sum_{i \in I} (\mathrm{card}\ A_{n-1} + 1)^{\varphi(i)} + 1$,

and if for an $i \in I$ there is $\lambda_A(\underbrace{\Lambda, \ldots, \Lambda}_{\varphi(i)}) = (b_1, \ldots, b_n) \in Q_n A \subseteq MFA$, then $\{ b_1, \ldots, b_n \} \subseteq A \setminus A_n$.

For any $i \in I$ define $f_i : (A \cup \{ \Lambda \})^{\varphi(i)} \longrightarrow \exp A$ by $f_i(a_1, \ldots, a_{\varphi(i)}) = \{ b_1, \ldots, b_n \}$, if $\lambda_A(a_1, \ldots, a_{\varphi(i)}) = (b_1, \ldots, b_n) \in Q_n A \subseteq MFA$, $f_i(a_1, \ldots, a_{\varphi(i)}) = \emptyset$ otherwise.

Choose: $x_0, y_0 \in A_n \setminus \bigcup_{i \in I} \bigcup \{ f_i(a);\ a \in (A_{n-1} \cup \{ \Lambda \})^{\varphi(i)} \}$, $x_0 \neq y_0$;

$x_1, y_1 \in A_{n-1} \setminus \bigcup_{i \in I} \bigcup \{ f_i(a);\ a \in (A_{n-2} \cup \{ \Lambda, x_0, y_0 \})^{\varphi(i)} \}$, $x_1 \neq y_1$;

$x_2 \in A_{n-2} \setminus \bigcup_{i \in I} \bigcup \{ f_i(a);\ a \in (A_{n-3} \cup \{ \Lambda, x_0, y_0, x_1, y_1 \})^{\varphi(i)} \}$;

$x_j \in A_{n-j} \setminus \bigcup_{i \in I} \bigcup \{ f_i(a);\ a \in (A_{n-j-1} \cup \{ \Lambda, x_0, y_0, x_1, y_1, x_2, x_3, \ldots, x_{j-1} \})^{\varphi(i)}$ for $j = 3, \ldots, n-1$.

II. Now, we prove the following assertion:

(i) Each of the elements $a = (x_0, x_1, x_2, \ldots, x_{n-1})$, $b = (x_0, y_1, x_2, \ldots, x_{n-1})$, $c = (y_0, x_1, x_2, \ldots, x_{n-1})$, $d = (y_0, y_1, x_2, \ldots, x_{n-1})$ occurs exactly once in the word

$$\lambda_A(x_0y_0, x_1y_1, x_2, \ldots, x_{n-1}) \in MFA.$$

(ii) Each of the elements a,b (a,c resp.) occurs exactly once in the word

$$\lambda_A(x_0, x_1y_1, x_2, \ldots, x_{n-1})$$

$$(\lambda_A(x_0y_0, x_1, x_2, \ldots, x_{n-1}) \text{ resp.}).$$

<u>Proof</u>. (i) Let $z = (z_0, z_1, \ldots, z_{n-1}) \in \{x_0, y_0\} \times \{x_1, y_1\} \times \{x_2\} \times \ldots \times \{x_{n-1}\}$.

Define $\alpha_z : A \longrightarrow MA$ by

$$\alpha_z(z_j) = z_j \text{ for } j = 0, \ldots, n-1$$

$$\alpha_z(x) = \wedge \text{ for } x \neq z_j.$$

Then according to the first distributive law,

$$\lambda_A F(\alpha^{\#}_z)(x_0y_0, x_1y_1, x_2, \ldots, x_{n-1}) = z \in MFA,$$

and according to the second distributive law,

$$z = (\lambda_A F(\alpha_z))^{\#} \lambda_A(x_0y_0, x_1y_1, x_2, \ldots, x_{n-1}).$$

Let $\lambda_A(x_0y_0, x_1y_1, x_2, \ldots, x_{n-1}) = u_1 \ldots u_k \in MFA$. From

$$(\lambda_A F(\alpha_z))^{\#}(u_1 \ldots u_k) = z \in FA \subseteq MFA$$

follows that there is exactly one $j \in \{1, \ldots, k\}$ such that $\lambda_A F(\alpha_z)(u_j) \neq \wedge$, $\lambda_A F(\alpha_z)(u_j) = z$.

Let $u_j = (v_1, \ldots, v_s) \in Q_s A \subseteq FA$.

There are two possibilities:

(a) $\{v_1, \ldots, v_s\} \subseteq \{z_0, \ldots, z_{n-1}\}$

(b) $\{v_1, \ldots, v_s\} \setminus \{z_0, \ldots, z_{n-1}\} \neq \emptyset$.

In the case (a) there is

$$\lambda_A F(\alpha_z)(u_j) = \lambda_A(u_j) = (v_1, \ldots, v_s) \in Q_s A \subseteq MFA$$

and necessarily $s = n$, $(v_1, \ldots, v_s) = (z_0, \ldots, z_{n-1})$.

In the case (b) there is

$F(\alpha_z)(u_j) = (t_1,\dots,t_s) \in Q_s(A \cup \{\Lambda\}) \subseteq FMA$ and $\Lambda \in \{t_1,\dots$
$\dots,t_s\}$.

It is evident that

$J = \{j \in \{0,\dots,n-1\};\ x_j \neq t_p$ for $p = 1,\dots,s$, and if $j \leq 1$ also $y_j \neq t_p$ for $p = 1,\dots,s\} \neq \emptyset$.

Suppose $j \in J$, $s = \varphi(i)$; $\lambda_A(t_1,\dots,t_s) = z$ is a word of length 1 and therefore $\lambda_A(t_1,\dots,t_s) = z = (z_0,\dots,z_{n-1})$, $\{z_0,\dots,z_{n-1}\} = f_i(t_1,\dots,t_s) \in \bigcup_{i \in I} \{f_i(a);\ a \in (A_{n-j-1} \cup \cup \{\Lambda, x_0, y_0, x_1, y_1, x_2, \dots, x_{j-1}\})^{\varphi(i)}\}$ which contradicts the assumption $z \in \{x_0,y_0\} \times \{x_1,y_1\} \times \{x_2\} \times \dots$
$\dots \times \{x_{n-1}\}$.

(ii) The proof is analogous.

III. Now, we can finish the proof of Proposition. We can assume without loss of generality that $(x_0,x_1,\dots,x_{n-1})$ is the first element of the set

$$\{x_0,y_0\} \times \{x_1,y_1\} \times \{x_2\} \times \dots \times \{x_{n-1}\}$$

which occurs in the word

$$\lambda_A(x_0y_0, x_1y_1, x_2, \dots, x_{n-1}).$$

(I.e. $\lambda_A(x_0y_0,x_1y_1,x_2,\dots,x_{n-1}) = \dots (x_0,x_1,\dots,x_{n-1}) \dots$
$\dots (x_0,y_1,x_2,\dots,x_{n-1}) \dots = \dots (x_0,x_1,\dots,x_{n-1}) \dots$
$\dots (y_0,x_1,\dots,x_{n-1}) \dots$)

Define $\alpha : A \longrightarrow MA$ by

$\alpha(x_0) = x_0y_0$,

$\alpha(y_0) = \Lambda$,

$\alpha(x) = x$ otherwise.

From the second distributive law and from II (ii) it follows that the element $(y_0,x_1,\dots,x_{n-1})$ occurs in the word

$$\lambda_A(x_0y_0, x_1y_1, x_2, \dots, x_{n-1})$$

before the element $(x_0, y_1, x_2, \dots, x_{n-1})$.

(I.e. $\lambda_A(x_0y_0, x_1y_1, x_2, \dots, x_{n-1}) = \dots (x_0, x_1, \dots, x_{n-1}) \dots$
$\dots (y_0, x_1, \dots, x_{n-1}) \dots (x_0, y_1, x_2, \dots, x_{n-1}) \dots$)

Define $\alpha' : A \longrightarrow MA$ by

$$\alpha'(x_1) = x_1y_1,$$
$$\alpha'(y_1) = \Lambda,$$
$$\alpha'(x) = x \text{ otherwise.}$$

By a similar reason, the element $(x_0, y_1, \dots, x_{n-1})$ occurs in the word $\lambda_A(x_0y_0, x_1y_1, x_2, \dots, x_{n-1})$ before the element $(y_0, x_1, \dots, x_{n-1})$.

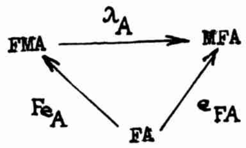
This contradiction finishes the proof of Proposition.

1.2. Corollary. Q_2 cannot be extended to the Kleisli category of Mon.

1.3. Proposition. $F = \bigvee_{n=1}^{+\infty} Q_n$ satisfies distributive laws over Mon.

Proof. Let A be a set. Define $\lambda_A : FMA \longrightarrow MFA$ by
$\lambda_A(x_{11} \dots x_{1k_1}, \dots, x_{n1} \dots x_{nk_n}) = (x_{11}, x_{12}, \dots, x_{nk_n}) \in Q_{k_1 + \dots + k_n}A \subseteq FA \subseteq MFA$ for $k_1 + \dots + k_n > 0$,
$\lambda_A(\Lambda, \dots, \Lambda) = \Lambda$.

(i)



commutes because

$$\lambda_A \cdot F(e_A)\underbrace{(x_1, \dots, x_n)}_{\in FA} = \lambda_A\underbrace{(x_1, \dots, x_n)}_{\in FMA} = \underbrace{(x_1, \dots, x_n)}_{\in FA \subseteq MFA} =$$
$$= e_{FA}(x_1, \dots, x_n).$$

(ii)

$$\begin{array}{ccc} FMA & \xrightarrow{\lambda_A} & MFA \\ {\scriptstyle F(\alpha^{\#})}\downarrow & & \downarrow{\scriptstyle (\lambda_B . F(\alpha))^{\#}} \\ FMB & \xrightarrow{\lambda_B} & MFB \end{array}$$

commutes for any $\alpha : A \longrightarrow MB$ because

$(\lambda_B F(\alpha))^{\#}\ \lambda_A(x_{11}\dots x_{1k_1},\dots,x_{n1}\dots x_{nk_n}) =$

$= (\lambda_B F(\alpha))^{\#}\ (x_{11},\dots,x_{nk_n}) = (\lambda_B F(\alpha))(x_{11},\dots,x_{nk_n}) =$

$= \lambda_B(y_{11}^{(1)}\dots y_{11}^{(m_{11})},\dots,y_{nk_n}^{(1)}\dots y_{nk_n}^{(m_{nk_n})}) =$

$= (y_{11}^{(1)}, y_{11}^{(2)},\dots,y_{nk_n}^{(m_{nk_n})})$ where $\alpha(x_{ij}) = y_{ij}^{(1)}\dots y_{ij}^{(m_{ij})}$,

and $\lambda_B F(\alpha^{\#})(x_{11}\dots x_{1k_1},\dots,x_{n1}\dots x_{nk_n}) =$

$= \lambda_B(y_{11}^{(1)}\dots y_{1k_1}^{(m_{1k_1})},\dots,y_{n1}^{(1)}\dots y_{nk_n}^{(m_{nk_n})}) =$

$= (y_{11}^{(1)}, y_{11}^{(2)},\dots,y_{nk_n}^{(m_{nk_n})})$;

obviously $(\lambda_B F(\alpha))^{\#}\ \lambda_A(\Lambda,\dots,\Lambda) = \Lambda = \lambda_B F(\alpha^{\#})(\Lambda,\dots,\Lambda)$.

This finishes the proof.

2.1. <u>Remark</u>. The propositions presented show that it is not so easy to decide whether a functor satisfies distributive laws, or not. The question is open even for sums of Q_n's and the monad Mon.

Define, for a moment, a "suitable" subset of $\underline{N}$ by the following equivalence: S is "suitable" iff $\coprod_{n \in S} Q_n$ satisfies distributive laws over Mon. It follows from [1] and from Propositions 1.1 and 1.3 that $\{1\}$ and $\underline{N}$ are "suitable", but every bounded subset of $\underline{N}$ which is not equal to $\{1\}$ is not "suitable".

2.2. Problem. Characterize all the "suitable" subsets of $\underline{N}$.

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Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, 18600 Praha 8
Československo

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