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ONE OBSTRUCTION FOR CLOSEDNESS

Jiří ROSICKÝ, Brno

Abstract: There is stated one necessary condition for a category to be closed. Applications are given to categories of algebras.

Key words: Closed category, adjoint on the right, variety of algebras, semigroup.

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This note deals with symmetric monoidal closed categories in the sense of [1]. In what follows, we will call them briefly closed categories. It is well-known (see [3]) that a variety $\mathbb{V}$ of universal algebras carries the structure of a closed category with the free algebra on one generator as the unit if and only if it is commutative. It means that for each n-ary operation f and for each algebra $\mathbb{V}$ the map $f: A^n \longrightarrow A$ is a homomorphism, i.e. for any m-ary operation $g: A^m \longrightarrow A$ it holds $gf^m = fg^n$. However, any variety of unary algebras is cartesian closed and thus there are non-commutative closed varieties. Recently, Foltz and Lair have deduced in [2] one necessary condition for closedness and they have shown that the variety of groups is not closed. We will give another obstruction for closedness which shows (in the same way as the criterion of

[2]) that the varieties of grupoids with unit and rings are not closed. The author is indebted to Ch. Lair and A. Pultr for a valuable conversation.

1. <u>The obstruction theorem</u>. Let $\mathbf{V}$ be a category. In what follows, we will suppose that E is an object of $\mathbf{V}$ such that every object of $\mathbf{V}$ is an iterated colimit of copies of E. It implies that the functor $/\ / = \mathbf{V}(E,-): \mathbf{V} \longrightarrow \mathrm{Set}$ is faithful.

<u>Definition 1</u>: A couple (F,φ) will be called a connection on a category $\mathbf{V}$ if $F: \mathbf{V}^{op} \longrightarrow \mathbf{V}$ is a functor and $\varphi = \varphi_{V,W}: \mathbf{V}(V,FW) \longrightarrow \mathbf{V}(W,FV)$ a natural isomorphism such that $\varphi_{W,V} \cdot \varphi_{V,W} = 1$ for any $V,W \in \mathbf{V}$.

A morphism of connections $\alpha: (F,\varphi) \longrightarrow (G,\gamma)$ is a natural transformation $\alpha: F \longrightarrow G$ such that $\gamma_{V,W} \cdot \mathbf{V}(V,\alpha_W) = \mathbf{V}(W,\alpha_V) \cdot \varphi_{V,W}$ for any $V,W \in \mathbf{V}$. In this way we get the category $C(\mathbf{V})$ of connections on $\mathbf{V}$.

If (F,φ) is a connection on $\mathbf{V}$, then F is adjoint on the right with itself, where φ is the adjunction isomorphism. Our terminology is an adaptation for our purposes of the terminology of Isbell [4], where a connection from a category A to a category B is a contravariant functor F from A to B having an adjoint on the right G. Under certain suppositions such connections can be identified with A-objects in B. Choosing objects $a \in A$ and $b \in B$ one gets underlying objects $Fa \in B$ and $Gb \in A$. So connections in our sense can be identified with double $\mathbf{V}$-objects (i.e. with $\mathbf{V}$-objects in $\mathbf{V}$) such that both underlying objects are isomorphic. Later, such an underlying object will be called pseudocommutative.

Denote by $\widetilde{\mathbb{V}}$ the category having objects (V,t), where $t: /V/ \longrightarrow /V/$ is a mapping such that $t^2 = 1$ and morphisms $f: (V,t) \longrightarrow (V',t')$ where $f: V \longrightarrow V'$ is a morphism in $\mathbb{V}$ and $t'/f/ = /f/t$. Define a functor $U: C(\mathbb{V}) \longrightarrow \widetilde{\mathbb{V}}$ by

$$U(F,\varphi) = (FE, \varphi_{E,E}) \text{ and } U\alpha = \alpha_E.$$

The following proposition is, in fact, a corollary of Theorem 3.8 from [4].

<u>Proposition 1</u>: U is full and faithful.

Proof: U is faithful following the property of E and the fact that connections take colimits to limits. If $f: U(F,\varphi) \longrightarrow$ $\longrightarrow U(G,\gamma)$ is a morphism in $\widetilde{\mathbb{V}}$, then

$$\beta_X: \mathbb{V}(E,FX) \xrightarrow{\varphi_{E,X}} \mathbb{V}(X,FE) \xrightarrow{\mathbb{V}(X,f)} \mathbb{V}(X,GE) \xrightarrow{\gamma^{-1}_{E,X}} \mathbb{V}(E,GX)$$

are components of a natural transformation $\beta: /F/ \longrightarrow /G/$. Following the proof of Theorem 3.8 of [4] f can be extended by colimits to a natural transformation $\alpha: F \longrightarrow G$ such that $\alpha_E = f$ and $/\alpha/ = \beta$. It immediately implies that $\gamma_{E,W} \cdot \mathbb{V}(E,\alpha_W) = \mathbb{V}(W,\alpha_E) \cdot \varphi_{E,W}$. Since $\mathbb{V}(W,G-)$ takes colimits and every object of $\mathbb{V}$ is an iterated colimit of copies of E, α have to be a morphism of connections.

<u>Definition 2</u>: An object V of $\mathbb{V}$ will be called pseudocommutative (with respect to E) if there is a connection (F,φ) on $\mathbb{V}$ such that $V \cong FE$.

So, a pseudocommutative object V is, under certain suppositions on $\mathbb{V}$, and underlying object of a double $\mathbb{V}$-object such that the second underlying object is isomorphic to V via isomorphism t such that $t^2 = 1$. Clearly, an object isomorphic to a pseudocommutative one is pseudocommutative.

Proposition 2: Let $\mathbb{V}$ have products. Then a product of pseudocommutative objects is pseudocommutative.

Proof: Let (F_i, φ_i) be connections on $\mathbb{V}$. Clearly $F = (\prod_{i \in I} F_i, \prod_{i \in I} \varphi_i)$ is their product, which implies the assertion.

Let $\mathbb{V}$ be a closed category. Denote by $\underline{\mathbb{V}}(-,-):$ $: \mathbb{V}^{op} \times \mathbb{V} \longrightarrow \mathbb{V}$ the internal hom-functor, by $- \otimes -:$ $\mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ the tensor product, by $p = p_{V,W,X}:$ $: \mathbb{V}(V \otimes W, X) \longrightarrow \mathbb{V}(V, \underline{\mathbb{V}}(W,X))$ the adjunction isomorphism, by $c = c_{V,W}: V \otimes W \longrightarrow W \otimes V$ the symmetry and by $I \in \mathbb{V}$ the unit. Then $(\underline{\mathbb{V}}(-,X), p_{W,V,X} \cdot \mathbb{V}(c,X) \cdot p^{-1}_{V,W,X})$ is a connection on $\mathbb{V}$ for any $X \in \mathbb{V}$.

Theorem 1: Let $\mathbb{V}$ be a closed category with products. Then any object of $\mathbb{V}$ is isomorphic to a subobject of a pseudocommutative object.

Proof: Since I is an iterated colimit of copies of E, $\mathbb{V}(I,V)$ is an iterated limit of copies of $\mathbb{V}(E,V)$ for any $V \in \mathbb{V}$. Using the construction of limits by products and equalizers and Proposition 2 we get that $\mathbb{V}(I,V)$ is a subobject of a pseudocommutative object. But V is isomorphic to $\mathbb{V}(I,V)$.

Remark: 1) The same assertion holds more generally for symmetric (non-monoidal) closed categories. These are (non-monoidal) closed categories endowed with a symmetry $s =$ $= s_{V,W,Z}: \mathbb{V}(V, \underline{\mathbb{V}}(W,Z)) \longrightarrow \mathbb{V}(W, \underline{\mathbb{V}}(V,Z))$ satisfying appropriate axioms (see [5]).

2) If $\mathbb{V}$ is an (epi-extremally mono)-category, then the word subobject can be replaced in Theorem 1 by an extremal subobject.

Another obstruction for closedness is stated in [2]. Roughly speaking, it asserts that if $\mathbb{V}$ is closed and any double $\mathbb{V}$-object underlies a triple one, then any object of $\mathbb{V}$ underlies a double one. This result can be strengthened in the sense that if any pseudocommutative object underlies a triple one, then any object underlies a double one.

2. Closed varieties of algebras. Let $\mathbb{V}$ be a variety of universal algebras and let E be the free algebra on one generator. Then / / is the usual forgetful functor.

Proposition 3: An algebra $V \in \mathbb{V}$ is pseudocommutative if and only if there is a bijection $t: /V/ \longrightarrow /V/$ such that $t^2 = 1$ and for any n-ary algebraic operation $h: /V/^n \longrightarrow /V/$ the mapping $tht^n: /V/^n \longrightarrow /V/$ is a homomorphism.

Clearly, tht^n are algebraic operations of a new algebra on /V/, which is isomorphic to V via t.
Denote by $Ps(\mathbb{V})$ the full subcategory of $\tilde{\mathbb{V}}$ consisting of objects (V,t) such that t makes V to be pseudocommutative. Then $Ps(\mathbb{V})$ is a new variety which arises from $\mathbb{V}$ by adding a new unary operation t and the axioms given by Proposition 3. Hence the forgetful functor $H: Ps(\mathbb{V}) \longrightarrow \mathbb{V}$ has a left adjoint L.

Corollary 1: Let the variety $\mathbb{V}$ admit a structure of a closed category. Then the unit η of the adjunction $L \dashv H$ is a monomorphism.

Proof: Since an object of $\mathbb{V}$ isomorphic to a objects from $Ps(\mathbb{V})$ belongs to $Ps(\mathbb{V})$, by Theorem 1 there is a monomorphism $V \xrightarrow{f} HV$. Hence η_V is mono because it factorizes through f.

Since $Ps(\mathbb{V})$ is a subvariety of $\tilde{\mathbb{V}}$, L is a suitable quotient of the left adjoint to the forgetful functor $\tilde{\mathbb{V}} \to$ $\to \mathbb{V}$. This left adjoint assigns to each $V \in \mathbb{V}$ and object $(W,t) \in \tilde{\mathbb{V}}$ where the underlying set $/W/$ of W is the union of a chain $X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$. Here $X_0 = /V/$, X_1 is the coproduct $/V/ \sqcup /V/$, X_2 is the underlying set of an algebra in $\mathbb{V}$ free over X_1 but with the algebraic structure of V on X_0, $X_3 = X_2 \sqcup (X_2 - X_1)$ and so on. The algebraic structure of W is clear and t is given by: t/X_1 interchanges the both copies of $/V/$, $t/X_3 - X_1$ interchanges the both copies of $X_2 - X_1$ and so on. This procedure is caused by the fact that $\tilde{\mathbb{V}}$ is the pullback in CAT of $\mathbb{V} \longrightarrow$ Set and of the forgetful functor of the category of algebras with the one unary operation t such that $t^2 = 1$.

L yields one general construction of pseudocommutative algebras. Another one is given by the following lemma.

Lemma 1: Let V_1 and V_2 be two algebras of $\mathbb{V}$ on the same underlying set X such that operations of V_1 are homomorphisms of V_2 (i.e. (V_1,V_2) is a double $\mathbb{V}$-object). Then t: $: X \times X \longrightarrow X \times X$, $t(x_1,x_2) = (x_2,x_1)$, makes $V_1 \times V_2$ to be pseudocommutative.

Proof is straightforward.

Example 1 (see [2]): Let $\mathbb{V}$ be the variety of groups.

It is well-known that double groups are commutative. Thus $\mathbb{V}$ is not closed. The same argument applies to the variety of semigroups (or grupoids resp.) with the unit.

Example 2: Let $\mathbb{V}$ be the variety of rings. It is easy to show that any double ring has the zero multiplication. Thus $\mathbb{V}$ is not closed.

Example 3: Let $\mathbb{V}$ be the variety of semigroups. A semigroup V is pseudocommutative iff there is a bijection t:
: $/V/ \longrightarrow /V/$ such that $t^2 = 1$ and $t(t(xy)t(x'y')) =$
$= t(t(x)t(x'))t(t(y)t(y'))$ for any $x,y,x',y' \in V$. Since the
left semigroup on X forms a double semigroup with any semi-
group on X, by Lemma 1 $/V \times V/$ carries the following pseudo-
commutative semigroup V^* for any semigroup V: $(u_1,u_2)(v_1,v_2) =$
$= (u_1v_1,u_2)$. Evidently V is a subsemigroup of V^* and thus we
do not need Corollary 1 for the verification that $\mathbb{V}$ overco-
mes Theorem 1. One is tempted to try what gives the assign-
ment $\underline{\mathbb{V}}(E,V) = V^*$. This functor $\underline{\mathbb{V}}(E,-): \mathbb{V} \longrightarrow \mathbb{V}$ has a
left adjoint $- \otimes E$ given as follows. Let $\approx$ be the transi-
tive hull of the relation $\sim$ on /V/ such that $u \sim v$ iff the-
re are $w, w_1, w_2 \in V$ such that $u = ww_1$, $v = ww_2$. Then $V \otimes E$ is
the coproduct of V and of a free semigroup on $/V//\approx$. Sin-
ce $V \otimes E = E$ for no semigroup V, we do not get a closed stru-
cture on $\mathbb{V}$. The author does not know whether such a struc-
ture can exist.

R e f e r e n c e s

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Přírodovědecká fakulta UJEP

Janáčkovo nám. 2a

662 95 Brno

Československo

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