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FIXED POINT THEOREMS FOR PSEUDOCONTRACTIVE MAPPINGS AND A COUNTEREXAMPLE FOR COMPACT MAPS

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Abstract: We give examples for open bounded starshaped sets in all normable spaces of dimension at least 3 whose closures have not the fixed point property for compact self-mappings. Using a special convergence theorem we extend fixed point theorems for pseudocontractive mappings (including nonexpansive mappings) which are known for Hilbert spaces.

Key words: Fixed points, starshaped sets, compact mappings, pseudocontractive and nonexpansive mappings, duality mappings.

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0. Introduction. It is well-known that the Brouwer fixed point theorem need not be true for compact starshaped subsets of a finite-dimensional space (see [12],[13],[14],[15]). However, the counterexamples given with respect to this problem are essentially boundary sets in the underlying space. In this paper we shall give even an example of a compact starshaped subset of three-dimensional space $\mathbb{R}^3$ which is the closure of an open starshaped set but has not the fixed point property for continuous maps. Moreover we shall present a theorem on (strong) convergence for Banach spaces having a weakly continuous duality mapping. Then we have both a generalization of a corresponding result for Hilbert spaces which is due to

M.G. Crandall and A. Pazy [3] and a couple of applications to the fixed point theory of pseudocontractive and nonexpansive mappings in Banach spaces possessing a weakly continuous duality mapping (for Hilbert spaces some of the results are known, see [12],[13],[14],[15],[16],[18]).

For a normed linear space $(E, \|\cdot\|)$ E^* denotes the strong dual space of $(E, \|\cdot\|)$ and for a subset X of E let $\overline{X}$, $\mathrm{int}(X)$, ∂X denote the closure of X, the interior of X and the boundary of X respectively. $X \subset E$ is said to be starshaped iff there exists $x_0 \in X$ such that $tx + (1-t)x_0 \in X$ for $x \in X$ and $t \in [0,1]$. For $f\colon X \longrightarrow E$ we define $\mathrm{Fix}(f) := \{x \mid x \in X \wedge f(x) = x\}$.

1. <u>A counterexample</u> In this section we give an example for an open bounded and starshaped subset of $\mathbb{R}^3$ whose closure has not the fixed point property for continuous self-mappings. Moreover we discuss some consequences of this result to $\mathbb{R}^n$ $(n \geq 3)$ and other spaces.

For the definition of the set in $\mathbb{R}^3$ described below we use a construction and a hint of J.M. Lysko [10].

<u>Theorem 1.1</u> There exist $X \subset \mathbb{R}^3$ and $f \in C(\overline{X}, \overline{X})$ such that

(i) X is open bounded and starshaped,

(ii) $\mathrm{Fix}(f) = \emptyset$.

<u>Proof</u>: (1) Let $p\colon [\frac{1}{2}, 1) \longrightarrow \mathbb{R}^+$ be defined as follows:

$$\mathop{\forall}_{\substack{n \in \mathbb{N} \\ n \geq 2}} \mathop{\forall}_{t \in [0,1]} p\Big((1-t)(1-\frac{1}{n}) + t(1-\frac{1}{n+1})\Big) := (n-2+t)\cdot\pi$$

Define X, $Y \subset \mathbb{R}^3$ respectively by

$$X := \{(r\cos\varphi, r\sin\varphi, z) \mid r \in [0,1),\ \varphi \in \mathbb{R}^+,\ z \in (-1,1)\} \cup$$
$$\{(|z| r\cos\varphi, |z| r\sin\varphi, z) \mid r \in (\tfrac{1}{2}, 1),\ 1 \leq |z| < 2,\ p(r) < \varphi < p(r) + 1\}$$

and

$$Y := \{(r\cos\varphi,\ r\sin\varphi, z) \mid r \in [0,1],\ \varphi \in \mathbb{R}^+,\ z \in [-1,1]\} \cup$$
$$\{(|z|\cos\varphi,\ |z|\sin\varphi, z) \mid 1 \leq |z| \leq 2,\ \varphi \in \mathbb{R}^+\} \cup$$
$$\{(|z|\, r\cos\varphi, |z| r\sin\varphi, z) \mid r \in [\tfrac{1}{2}, 1),\ 1 \leq |z| \leq 2,$$
$$p(r) \leq \varphi \leq p(r) + 1\}.$$

By a straightforward but somewhat lengthy computation we obtain: X is a bounded open and starshaped (with respect to the origin) subset of $\mathbb{R}^3$ such that $\overline{X} = Y$. Now let $H: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined as follows:

If $(x,y,z) \in \mathbb{R}^3$ and $|z| \leq 1$ then $H(x,y,z) := (x,y,z)$;
if $(x,y,z) \in \mathbb{R}^3$ and $|z| > 1$ then $H(x,y,z) := (\frac{x}{|z|}, \frac{y}{|z|}, z)$.

H clearly is a homeomorphism. Let $K := H[Y]$. Thus

$$K = \{(r\cos\varphi, r\sin\varphi, z) \mid r \in [0,1],\ \varphi \in \mathbb{R}^+,\ |z| \leq 1\} \cup$$
$$\{(\cos\varphi, \sin\varphi, z) \mid \varphi \in \mathbb{R}^+,\ 1 \leq |z| \leq 2\} \cup$$
$$\{(r\cos\varphi, r\sin\varphi, z) \mid r \in [\tfrac{1}{2}, 1),\ 1 \leq |z| \leq 2,\ p(r) \leq$$
$$\leq \varphi \leq p(r) + 1\}.$$

Clearly it is enough to prove the existence of a $g \in C(K,K)$ such that $\mathrm{Fix}(g) = \emptyset$.

Let $R: K \longrightarrow \mathbb{R}^3$ be defined as follows:

If $(x,y,z) \in K$ and $z \leq 1$ then $R(x,y,z) := (x,y,1)$;
if $(x,y,z) \in K$ and $x^2 + y^2 = 1$ and $z \geq 1$ then $R(x,y,z) := (x,y,z)$;
if $(x,y,z) \in K$ and $x = r\cos\varphi$, $y = r\sin\varphi$ and $r \in [\frac{1}{2}, 1), z \geq 1$ and $p(r) \leq \varphi \leq p(r) + 1$ then

$$R(x,y,z) := \begin{cases} (p^{-1}(\varphi)\cos\varphi, p^{-1}(\varphi)\sin\varphi, z + r - p^{-1}(\varphi)) \\ \qquad\qquad\qquad\qquad \text{if } z + r - p^{-1}(\varphi) \geq 1 \\ ((z - 1 + r)\cos\varphi,\ (z - 1 + r)\sin\varphi, 1) \\ \qquad\qquad\qquad\qquad \text{if } z + r - p^{-1}(\varphi) < 1. \end{cases}$$

For $z \in [1,2]$ and $r \in [\frac{1}{2}, 1)$ such that $p(r) \leq \varphi \leq p(r) + 1$

we have $z + r - p^{-1}(\varphi) \leq z \leq 2$ and conversely $z + r - p^{-1}(\varphi) \leq$ ≤ 1 implies

$\frac{1}{2} \leq r \leq z - 1 + r \leq p^{-1}(\varphi) < 1$ thus

$$R[K] \subset K' := \{(r\cos\varphi, r\sin\varphi, 1) \mid r \in [0,1],\ \varphi \in \mathbb{R}^+\} \cup \{(\cos\varphi, \sin\varphi, z) \mid z \in [1,2],\ \varphi \in \mathbb{R}^+\} \cup \{(p^{-1}(\varphi)\cos\varphi, p^{-1}(\varphi)\sin\varphi, z) \mid z \in [1,2],\ \varphi \in \mathbb{R}^+\} \subset K.$$

Moreover we have $R|K' = \mathrm{Id}_{K'}$, thus $R[K'] = K'$.
Obviously R is continuous (it clearly suffices to verify this for points $(\cos\varphi_0, \sin\varphi_0, z_0)$ with $\varphi_0 \in \mathbb{R}^+$ and $z_0 \in [1,2]$). We have: R is a retraction from K onto $K' \subset K$. Therefore it is enough to search for a map $h \in C(K', K')$ such that $Fix(h) = \emptyset$ (then $g := h \circ R \in C(K,K)$ and $Fix(g) = \emptyset$). Let $h: K' \longrightarrow \mathbb{R}^3$ be defined as illustrated in the schedule on page 296.

<u>Remark 1.2</u>. (i) Let $n \in \mathbb{N}$, let $X \subset \mathbb{R}^n$ be open bounded and starshaped with respect to $b \in X$. Suppose $f \in C(\overline{X}, \overline{X})$ such that $Fix(f) = \emptyset$. Let $j: \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ be the natural embedding. Then the "open cone over X"

$$Y := \{(z_1, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} \mid z_{n+1} \in (0,1) \wedge \frac{1}{1 - z_{n+1}}(z_1, \ldots, z_n) \in X\}$$

is an open bounded and starshaped subset of $\mathbb{R}^{n+1}$ and $g \in C(\overline{Y}, \overline{Y})$ defined by $g((1-t)(0,\ldots,0,1) + tj(x)) := j(f((1-t)b + tx))$ for $t \in [0,1]$, $x \in \overline{X}$ has no fixed points (compare [8]). Thus we obtain by Theorem 1.1:

$$\forall_{\substack{n \in \mathbb{N} \\ n \geq 3}} \ \exists_{X \subset \mathbb{R}^n} \ \exists_{f \in C(\overline{X}, \overline{X})} \ X \text{ open bounded starshaped} \wedge Fix(f) = \emptyset.$$

(ii) Let $(E, \mathcal{T})$ be a separated locally convex topological linear space of dimension at least 3. Then there exist an open starshaped set K and a compact map $g \in C(\overline{K},\overline{K})$ such that $Fix(g) = \emptyset$. Indeed, let F be a 3-dimensional linear subspace of E. As there are linear homeomorphisms between $\mathbb{R}^3$ and F, there is an open bounded starshaped set (with respect to the origin) X in F and $f \in C(\overline{X},\overline{X})$ with $Fix(f) = \emptyset$. Let P be any continuous linear projection of E onto F (F is a complementary set). Set $K := P^{-1}[X]$ and $g := f \circ P|\overline{K}$. Then it is clear that K is open and starshaped and g is a compact map such that $Fix(g) = \emptyset$. If, in addition, $(E,\mathcal{T})$ is normable, K may be taken bounded.

2. Fixed point theorems for pseudocontractive mappings.

A convergence theorem due to M.G. Crandall and A. Pazy [3] implies several fixed point theorems for continuous pseudocontractive and especially for nonexpansive mappings in Hilbert space (see [12],[13],[14],[15],[16],[18]). In the present note we establish a variant of that theorem which guarantees that most of these results are valid for a more general class of spaces.

Definition 2.1 (i) $\mu : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is said to be a gauge function $:\Longleftrightarrow$ μ is continuous and strictly monotone, $\mu(0) = 0$, $\lim\limits_{t\to\infty} \mu(t) = \infty$.

(ii) Let $(E, \|\cdot\|)$ be a real normed space, $\mu: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a gauge function and $J: E \longrightarrow E^*$. J is called a duality mapping with respect to $\mu :\Longleftrightarrow \forall_{x \in E}\ J(x)(x) = \|x\| \cdot \mu(\|x\|) \wedge \|J(x)\| = \mu(\|x\|)$.

(iii) $((E,\|\cdot\|), \mu, J)$ satisfies $(*) :\Longleftrightarrow (E,\|\cdot\|)$ is a reflexive real normed space, $\mu: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a gauge function and $J: E \longrightarrow E^*$ is a weakly sequentially continuous duality mapping with respect to μ [1].

<u>Remark 2.2</u> (i) Let $(E, \|\cdot\|)$ be a real normed space, let $\mu: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a gauge function. Then the Hahn-Banach theorem implies the existence of a duality mapping $J: E \longrightarrow E^*$ with respect to μ.

(ii) Let $(E,(\cdot,\cdot))$ be a real Hilbert space. Define $J: E \longrightarrow E^*$ by $J(x)(y) := (y,x)$ and $\|\cdot\|: E \longrightarrow \mathbb{R}$ by $\|x\| := (x,x)^{1/2}$. Then $((E, \|\cdot\|), \mathrm{Id}_{\mathbb{R}^+}, J)$ satisfies $(*)$.

(iii) Let $p, q \in (1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. Then we identify ℓ_p^*, ℓ_q in the usual manner. Define $J: \ell_p \longrightarrow \ell_q$ by $J((x_j)_{j\in\mathbb{N}}) := (|x_j|^{p-1} \operatorname{sign} x_j)_{j\in\mathbb{N}}$ and $\mu: \mathbb{R}^+ \longrightarrow \mathbb{R}$ by $\mu(t) := t^{p-1}$. Then $((\ell_p, \|\cdot\|), \mu, J)$ satisfies $(*)$. (See [2]).

<u>Definition 2.3</u> Let $(E, \|\cdot\|)$ be a normed space, $\emptyset \neq X \subset E$, $f: X \longrightarrow E$.

(i) f is said to be nonexpansive $:\Longleftrightarrow \underset{x,y\in X}{\forall} \|f(x) - f(y)\| \leq \|x - y\|$

(ii) f is said to be pseudocontractive $:\Longleftrightarrow$

$$\underset{x,y\in X}{\forall} \ \underset{r\in\mathbb{R}}{\forall} \|x - y\| \leq \|(1 + r)(x - y) - r\cdot(f(x) - f(y))\|$$

<u>Remark 2.4</u> Let $(E, \|\cdot\|)$ be a real normed space, $\emptyset \neq X \subset E$, $f: X \longrightarrow E$.

(i) If f is nonexpansive then f is pseudocontractive.

(ii) If there is a uniquely determined duality mapping

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1) This implies that E^* is strictly-convex (see [6]) and consequently J is unique.

$J: E \longrightarrow E^*$ with respect to some gauge function then we have: f pseudocontractive $\iff \forall_{x,y \in X} J(x - y)(f(x) - f(y)) \leq \leq J(x - y)(x - y)$ (see [7]).

The announced convergence theorem is

__Lemma 2.5__ ([12]) __Let__ $(E, \|\cdot\|)$ be a real normed space admitting a weakly sequentially continuous duality mapping $J: E \longrightarrow E^*$ with respect to some gauge function μ, let $(x_n) \in E^{Z^+}$, $(r_n) \in (0,\infty)^{\mathbb{N}}$ such that

(i) $\lim(x_n) = x_o$ (weakly), (ii) $\lim(r_n) = 0$,

(iii) $\forall_{n,m \in \mathbb{N}} J(x_n - x_m)(r_n x_n - r_m x_m) \leq 0$

__Then__ $\lim(x_n) = x_o$ (strongly).

__Proof__: We have for $n \in \mathbb{N}$ $\lim_m (J(x_n - x_m)) = J(x_n - x_o)$ (weakly), $\lim_m(-r_n x_n + r_m x_m) = -r_n x_n$ (strongly). This together with (iii) implies: $J(x_n - x_o)(-r_n x_n) = \lim_m (J(x_n - x_m)(-r_n x_n + r_m x_m)) \geq 0$, hence $J(x_n - x_o)(-x_o) = J(x_n - x_o)(x_n - x_o) + J(x_n - x_o)(-x_n) \geq J(x_n - x_o)(x_n - x_o) = \| x_n - x_o \| \cdot \mu(\| x_n - x_o \|)$. Because of $\lim(J(x_n - x_o)(-x_o)) = 0$ we get $\lim(\| x_n - x_o \|) = 0$.

As an evident consequence of Lemma 2.5 we get

__Lemma 2.6__ __Let__ $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, $(x_n) \in E^{\mathbb{N}}$, $(r_n) \in (0,\infty)^{\mathbb{N}}$ such that (x_n) is bounded, $\lim(r_n) = 0$ and $\forall_{n,m \in \mathbb{N}} J(x_n - x_m)(r_n x_n - r_m x_m) \leq 0$.

__Then__ there is a subsequence (y_n) of (x_n) and $y \in E$ such that $\lim(y_n) = y$ (strongly).

Lemma 2.6 implies the following fixed point theorem for continuous pseudocontractive mappings:

Lemma 2.7 (See [16]) Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, let $\emptyset \neq X \subset E$ be closed and $f: X \longrightarrow E$ be continuous and pseudocontractive, let $(x_n) \in X^{\mathbb{N}}$, $(\lambda_n) \in (0,1)^{\mathbb{N}}$ such that

(i) (x_n) is bounded, (ii) $\lim(\lambda_n) = 1$,

(iii) $\underset{m \in \mathbb{N}}{\forall}\ x_n = \lambda_n f(x_n)$

Then f has a fixed point.

Proof: We define $(r_n) \in (0,\infty)^{\mathbb{N}}$ by $r_n := \frac{1}{\lambda_m} - 1$. As f is pseudocontractive we get for $n, m \in \mathbb{N}$:

$-J(x_n - x_m)(r_n x_n - r_m x_m) = J(x_n - x_m)(x_n - x_m - f(x_n) + f(x_m)) \geq 0$ (see Remark 2.4 (ii)). Lemma 2.6 guarantees $y \in E$ and a subsequence (y_n) of (x_n) such that $\lim(y_n) = y$ (strongly).

Then $y \in X$ and because of $\lim(y_n - f(y_n)) = 0$ and continuity of f we get: $f(y) = y$.

The following theorems are applications of Lemma 2.7. For Hilbert spaces and lipschitzian pseudocontractive mappings the theorems 2.8 and 2.9 are proved in [16] and for merely continuous pseudocontractive mappings they are proved in [18].

Theorem 2.8 Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, let $X \subset E$ be a closed neighborhood of the origin and $f: X \longrightarrow E$ be continuous and pseudocontractive such that $f[X]$ is bounded and

(LS) $\underset{x \in \partial X}{\forall}\ \underset{\lambda \in \mathbb{R}}{\forall}\ f(x) = \lambda x \Longrightarrow \lambda \leq 1$

Then f has a fixed point.

Proof: Choose $(\lambda_n) \in (0,1)^{\mathbb{N}}$ with $\lim(\lambda_n) = 1$. For $n \in \mathbb{N}$ $\lambda_n f$ is continuous and strictly pseudocontractive with

$\underset{x \in \partial X}{\forall}\ \underset{\lambda \in \mathbb{R}}{\forall}\ (\lambda_n f)(x) = \lambda x \Longrightarrow \lambda \leq 1.$

By a theorem of R. Schöneberg [18] there is $(x_n) \in X^{\mathbb{N}}$ such that $x_n = \lambda_n f(x_n)$ for $n \in \mathbb{N}$. According to Lemma 2.7 we are done.

__Theorem 2.9__ __Let__ $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, let $X \subset E$ be a closed and symmetric neighborhood of the origin and $f: X \longrightarrow E$ be continuous and pseudocontractive such that $f[X]$ is bounded and $\forall_{x \in \partial X} f(-x) = -f(x)$.

__Then__ f has a fixed point.

__Proof__: For $x \in \partial X$ we have $J(2x)(2f(x)) = J(x - (-x)) \cdot (f(x) - f(-x)) \leq J(x - (-x))(x - (-x)) = J(2x)(2x)$. Thus f satisfies condition (LS) of Theorem 2.8.

__Lemma 2.10__ __Let__ E be a topological linear space and $X \subset E$ be starshaped with respect to the origin. Assume $f: \overline{X} \longrightarrow E$ such that

$$\text{(R)} \qquad \forall_{x \in \partial X} \ \exists_{\lambda > 0} \ \forall_{t \in (0, \lambda]} (1 + t)x - tf(x) \notin \overline{X}.$$

__Then__ $\forall_{x \in \partial X} \ \forall_{\lambda \in \mathbb{R}} \ f(x) = \lambda x \Longrightarrow \lambda \leq 1.$

__Proof__: Let $x \in \partial X$, $\lambda \in \mathbb{R}$ and $f(x) = \lambda x$. Suppose $\lambda > 1$. Choose $\tilde{\lambda} > 0$ such that $(1 + t)x - tf(x) \notin \overline{X}$ for $t \in (0, \tilde{\lambda}]$ and choose $t \in (0, \tilde{\lambda}]$ such that $(\lambda - 1)t \in (0, 1]$. Then we have $(1 + t)x - tf(x) = (1 - (\lambda - 1)t)x \in \overline{X}$ since $\overline{X}$ is starshaped with respect to the origin, too. This contradicts (R), thus $\lambda \leq 1$.

Observing Lemma 2.10 and Theorem 2.8 we obtain

__Theorem 2.11__ __Let__ $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$. Suppose $X \subset E$ is closed and starshaped with respect to $0 \in \operatorname{int}(X)$ and $f: X \longrightarrow E$ is continuous and pseudocontractive such that $f[X]$ is bounded and

(R) $\underset{x\in\partial X}{\forall}\ \underset{\lambda>0}{\exists}\ \underset{t\in(0,\lambda]}{\forall}\ (1+t)x - tf(x) \notin \overline{X}.$

Then f has a fixed point.

Remark 2.12 Lemma 2.10 shows that H. Rothe´s fixed point theorem for compact maps in [17] is only a special case of the general Leray-Schauder fixed point theorem for compact maps.

Theorem 2.13 Let $(E, \|\cdot\|), \mu, J)$ satisfy $(*)$. Suppose $X \subset E$ is a closed bounded and symmetric neighborhood of the origin, $f\colon X \longrightarrow E$ is continuous and pseudocontractive such that $f[X]$ is bounded and

(A) $\underset{\varepsilon>0}{\exists}\ \underset{x\in\partial X}{\forall} \|f(x) + f(-x)\|^2 - \|2x - f(x) + f(-x)\|^2 \leq$
$\leq 4(1-\varepsilon)\|x - f(x)\| \cdot \|x + f(-x)\|$

and

(B) $\inf\{\|x - f(x)\| \mid x \in \partial X\} > 0$

Then f has a fixed point.

Proof: Let $\varepsilon > 0$ be chosen according to (A). Let $M>0$ such that $\|f(x)\| < M$, $\|x\| < M$ for $x \in X$, $r := \inf\{\|x - f(x)\| \mid x \in \partial X\}$, let $(\lambda_n) \in (0,1)^{\mathbb{N}}$ such that $\lim(\lambda_n) = 1$ and $(1-\lambda_n)\cdot 9M^2 < \varepsilon\cdot r^2$ for $n \in \mathbb{N}$. Then we have for $x \in \partial X$, $n \in \mathbb{N}$:

$$\frac{1}{4}\|\lambda_n f(x) + \lambda_n f(-x)\|^2 - \frac{1}{4}\|2x - \lambda_n f(x) + \lambda_n f(-x)\|^2$$
$$\leq \frac{1}{4}\|f(x) + f(-x)\|^2 - \frac{1}{4}\|2x - f(x) + f(-x)\|^2 +$$
$$+ (1-\lambda_n)4M^2 \leq (1-\varepsilon)\|x - f(x)\| \cdot \|x + f(-x)\| +$$
$$+ (1-\lambda_n)\cdot 4M^2 \leq \|x - f(x)\| \cdot \|x + f(-x)\| - \varepsilon\cdot r^2 +$$
$$+ (1-\lambda_n)\cdot 4M^2 \leq (\|x - \lambda_n f(x)\| + (1-\lambda_n)M)\cdot$$

$\cdot(\| x + \lambda_n f(-x) \| + (1 - \lambda_n)M) - \varepsilon \cdot r^2 + (1 - \lambda_n) \cdot 4M^2$

$< \| x - \lambda_n f(x) \| \cdot \| x + \lambda_n f(-x) \|$, hence

$x - \lambda_n f(x) \neq \mu(-x - \lambda_n f(-x))$ for $n \in \mathbb{N}$, $x \in \partial X$,

$\mu \in (0,1]$.

By a theorem of R. Schöneberg [18] we obtain a sequence (x_n) such that $x_n = \lambda_n f(x_n)$. Hence f has a fixed point by Lemma 2.7.

Remark 2.14 (i) In the case of a Hilbert space $(E,(\cdot,\cdot))$ the condition (A) of Theorem 2.13 is equivalent to

$$\exists_{\varepsilon>0} \ \forall_{x \in \partial X} \left(\frac{x - f(x)}{\|x - f(x)\|} , \frac{-x-f(-x)}{\|-x - f(-x)\|} \right) \leq 1 - \varepsilon$$

(ii) For nonexpansive mappings we get the following

Theorem: Let $(E, \|\cdot\|)$ be a uniformly convex space. Suppose $X \subset E$ is a closed bounded convex symmetric neighborhood of the origin and let $f: X \longrightarrow E$ be nonexpansive such that (A) of Theorem 2.13 is fulfilled.

Then f has a fixed point.

The proof is based upon the fact that $Id_X - f$ is demiclosed.

Theorem 2.15 (see [18]) Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, let $X \subset E$ be closed and bounded with $\mathrm{int}(X) \neq \emptyset$. Suppose $f: X \longrightarrow E$ is continuous and pseudocontractive such that $f[X]$ is bounded and $\exists_{z \in X} \ \forall_{x \in \partial X} \| z - f(z) \| < \| x - f(x) \|$ ("minimum principle")

Then f has a fixed point.

Proof: Theorem 1 of [9] implies $\inf \{\| x - f(x)\| \mid x \in X\} = 0$. Without loss of generality we may assume that

$a := \inf\{\|x - f(x)\| \mid x \in \partial X\} > 0$ and that there exists $z \in X$ such that $\|z - f(z)\| < a$. Moreover we may assume $\|f(0)\| < a$. Choose $(r_n) \in (0,\infty)^{\mathbb{N}}$ such that $\lim(r_n) = 0$ and $r_n\|x\| + \|f(0)\| < a$ for $n \in \mathbb{N}$ and $x \in X$. Define $T_n\colon X \longrightarrow E$ by $T_n := (1 + r_n)Id_X - f$, let $n \in \mathbb{N}$. Then we have for $x, y \in X$:
$\mu(\|x - y\|)\|T_n(x) - T_n(y)\| \geq J(x - y)(T_n(x) - T_n(y)) \geq$
$\geq J(x - y)(r_n x - r_n y) = r_n \mu(\|x - y\|)\|x - y\|$, hence
(0) $\|T_n(x) - T_n(y)\| \geq r_n\|x - y\|$, and for $x \in \partial X$:
$\|T_n(0)\| = \|f(0)\| < \|x - f(x)\| - r_n\|x\| \leq \|T_n(x)\|$. Theorem 1 of [9] implies: $0 \in \overline{T_n[X]}$, and because of (0): $0 \in T_n[X]$. That means: There is $(x_n) \in X^{\mathbb{N}}$ such that $x_n = \frac{1}{1+r_n} f(x_n)$ for $n \in \mathbb{N}$. Lemma 2.7 completes the proof.

<u>Remark 2.16</u> From Theorem 1 of [9] we learn that Theorem 2.15 remains true if the assumption "$((E,\|\cdot\|),\mu,J)$ satisfies $(*)$" is replaced by "$(E,\|\cdot\|)$ is a Banach space and X has the fixed point property with respect to nonexpansive self-mappings".

<u>Lemma 2.17</u> <u>Let</u> $(E,\|\cdot\|)$ be a normed space. Suppose $X \subset E$ is closed and starshaped with respect to the origin, $\lambda \in (0,1)$ and $f\colon X \longrightarrow E$ such that

$$\forall_{x \in \partial X} \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} d((1 - t)x + tf(x), X) = 0$$

<u>Then</u> $$\forall_{x \in \partial X} \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} d((1 - t)x + t \cdot \lambda f(x), X) = 0$$

<u>Theorem 2.18</u> <u>Let</u> $((E,\|\cdot\|),\mu,J)$ satisfy $(*)$. Suppose $X \subset E$ is closed, bounded and starshaped and $f\colon X \longrightarrow E$ is continuous and pseudocontractive such that

$$\forall_{x \in \partial X} \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} d((1 - t)x + tf(x), X) = 0$$

Then f has a fixed point.

Proof: Define $\tilde{J}: E \longrightarrow E^*$ by $J(0) := 0$, $\tilde{J}(x) :=$
$:= \frac{\|x\|}{\mu(\|x\|)} J(x)$ for $x \in E \setminus \{0\}$. $\tilde{J}$ is the (uniquely determined) duality mapping with respect to $Id_{\mathbb{R}^+}$. Without loss of generality we assume X to be starshaped with respect to the origin. Choose $(\lambda_n) \in (0,1)^{\mathbb{N}}$ such that $\lim(\lambda_n) = 1$. Then we have for $n \in \mathbb{N}$:

(i) $\lambda_n f$ is continuous

(ii) $\tilde{J}(x - y)(\lambda_n f(x) - \lambda_n f(y)) \leq \lambda_n \|x - y\|^2$

(iii) $\forall_{x \in \partial X} \lim_{\substack{t \to 0 \\ t > 0}} \frac{1}{t} d((1 - t)x + t \cdot \lambda_n f(x), X) = 0$ (Lemma 2.17)

A theorem of R.H. Martin [11] and K. Deimling [4] implies the existence of $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ such that $x_n = \lambda_n f(x_n)$ for $n \in \mathbb{N}$, and Lemma 2.7 completes the proof.

Remark 2.19 For lipschitzian pseudocontractive mappings in Hilbert spaces Theorem 2.18 was proved by D. Göhde [5]. If X is assumed to be convex it was shown in [16] that the assumption "f be lipschitzian" can be dropped.

Theorem 2.20 **Let** $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$. Suppose $\emptyset \neq X \subset E$ is closed and bounded and $f: X \longrightarrow E$ is nonexpansive such that $\mathrm{co}\, f[\partial X] \subset X$.

Then f has a fixed point.

Proof: Without loss of generality $0 \in f[\partial X]$. Let $(\lambda_n) \in (0,1)^{\mathbb{N}}$ with $\lim(\lambda_n) = 1$. For $n \in \mathbb{N}$, $x \in \partial X$ we have: $\lambda_n f$ is a Banach-contraction and $(\lambda_n f)(x) \in \mathrm{co}\, f[\partial X] \subset X$, thus $(\lambda_n f)[\partial X] \subset X$.

According to a theorem due to N.A. Assad [1] there is $(x_n) \in X^{\mathbb{N}}$ such that $x_n = \lambda_n f(x_n)$ for $n \in \mathbb{N}$, and by Lemma 2.7 we obtain the conclusion.

Lemma 2.21 Let $(E, \|\cdot\|)$ be a normed space, $J: E \longrightarrow E^*$ be a duality mapping with respect to some gauge function $\mu: \mathbb{R}^+ \longrightarrow \mathbb{R}$. Suppose $x, z \in E$, $M > 0$, $\|x\| \geq 3M$, $\|z\| < M$.
Then (i) $0 < \mu(2M) \leq \|J(x - z)\| \leq \mu(\|x\| + M)$

(ii) $J(x - z)(x) \geq \mu(2M) \cdot M$.

Proof: (i) $\|J(x - z)\| = \mu(\|x - z\|) \geq \mu(\|x\| - \|z\|) \geq \mu(3M - M)$, $\|J(x - z)\| \leq \mu(\|x\| + \|z\|) \leq \mu(\|x\| + M)$

(ii) $J(x - z)(x) = J(x - z(x - z) + J(x - z)(z)$

$\geq \|J(x - z)\| (\|x - z\| - \|z\|) \geq \mu(2M) \cdot M$.

Lemma 2.22 Let $(E, \|\cdot\|)$ be a normed space, $M, r > 0$, $x \in E$, $\|x\| \geq 3M$, $\emptyset \neq S \subset E$ and suppose $\|z\| < M$ for $z \in S$.
Then $\inf_{z \in S} \|(1 + r)x - z\| > \inf_{z \in S} \|x - z\|$

Proof: Let $J: E \longrightarrow E^*$ be a duality mapping with respect to $Id_{\mathbb{R}^+}$. Then we have for $z \in S$: $\|J(x - z)\| \|(1 + r)x - z\| \geq J(x - z)(x - z) + J(x - z)(rx) = \|J(x - z)\| \|x - z\| + r J(x - z)(x) \geq \|J(x - z)\| \|x - z\| + r \cdot 2 \cdot M \cdot M$, and from

$$\|(1 + r)x - z\| \geq \|x - z\| + \frac{2rM^2}{\|J(x-z)\|} \geq \|x - z\| + \frac{2rM^2}{\|x\| + M}$$

and $\frac{2rM^2}{\|x\| + M} > 0$ the conclusion follows.

Lemma 2.23 Let $(E, \|\cdot\|)$ be a normed space, $\emptyset \neq X \subset E$ and $f: X \longrightarrow E$ be nonexpansive. Suppose $x_0 \in X$ such that $(f^n(x_0))_{n \in \mathbb{Z}^+}$ is bounded. Finally let $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ such that $f(x_n) = (1 + \frac{1}{n})x_n$ for $n \in \mathbb{N}$
Then $(x_n)_{n \in \mathbb{N}}$ is bounded.

Proof: Define $S := \{ f^n(x_o) \mid n \in \mathbb{Z}^+ \}$. Obviously $S \neq \emptyset$. Choose $M > 0$ such that $\| z \| < M$ for $z \in S$. We claim $\| x_n \| < 3M$ for $n \in \mathbb{N}$. Otherwise we would have $\| x_n \| \geq 3M$ for a suitable $n \in \mathbb{N}$, hence by Lemma 2.20 $\inf_{z \in S} \| f(x_n) - z \| = \inf_{z \in S} \| (1 + \frac{1}{n}) x_n - z \| > \inf_{z \in S} \| x_n - z \|$ for this n. Choose $y \in S$ with $\| x_n - y \| < \inf_{z \in S} \| f(x_n) - z \|$. Observing $f(y) \in S$ we get $\| f(x_n) - f(y) \| \geq \inf_{z \in S} \| f(x_n) - z \| > \| x_n - y \|$ and this is a contradiction to the nonexpansiveness of f.

Theorem 2.24 Let $((E, \| \cdot \|), \mu, J)$ satisfy $(*)$. Suppose $\emptyset \neq X \subset E$ is closed and starshaped and $f: X \longrightarrow E$ is nonexpansive such that:

(i) $f[\partial X] \subset X$

(ii) there is $x_o \in X$ such that $(f^n(x_o))_{n \in \mathbb{Z}^+}$ is bounded.

Then f has a fixed point.

Proof: Without loss of generality let X be starshaped with respect to the origin. Then for $n \in \mathbb{N}$ $(1 - \frac{1}{n+1}) f$ is a Banach-contraction with $(1 - \frac{1}{n+1}) f[\partial X] \subset X$. By a theorem of N.A. Assad [1] there is $(x_n) \in X^{\mathbb{N}}$ such that $x_n = (1 - \frac{1}{n+1}) f(x_n)$ for $n \in \mathbb{N}$. The boundedness of (x_n) follows from Lemma 2.23; Lemma 2.7 completes the proof.

Remark 2.25 (i) Theorem 2.24 was originally proved for Hilbert spaces by J. Reinermann and R. Schöneberg [16].

(ii) In the case of a Hilbert space $(E, (\cdot, \cdot))$ and a convex X Theorem 2.24 remains valid if condition (i) is cancelled.

	$h(x\cos\varphi, x\sin\varphi, 1)$ $\varphi\in\mathbb{R}^+$, $x\in[0,\frac{1}{2}]$	$h(r^{-1}(\varphi)\cos\varphi,\ r^{-1}(\varphi)\sin\varphi, x)$ $0\le\varphi\le\pi$	$h(r^{-1}(\varphi)\cos\varphi, r^{-1}(\varphi)\sin\varphi, x), \varphi\ge\pi$ $h(x\cos\varphi, x\sin\varphi, 1),\ 2/3\le x<1$	$h(\cos\varphi, \sin\varphi, x)$ $\varphi\in\mathbb{R}^+$
$x\in[3/2, 2]$		$(r^{-1}((\varphi+\pi)(x-1))\cos((\varphi+\pi)(x-1)), \ldots$ $\ldots, 2-\frac{\varphi}{\pi}(1-\frac{x}{2}))$	$(r^{-1}(\varphi-3\pi+2\pi x)\cos(\varphi-3\pi+2\pi x), \ldots$ $\ldots, 1+\frac{x}{2})$	$(\cos(\varphi-3\pi+2\pi x), \ldots$ $\ldots, 1+\frac{x}{2})$
$x\in[1, 3/2]$		$(r^{-1}((\varphi+\pi)(x-1))\cos((\varphi+\pi)(x-1)), \ldots$ $\ldots, 2-\frac{\varphi}{\pi}(\frac{5}{2}-\frac{3}{2}x))$	$(r^{-1}(\varphi-3\pi+2\pi x)\cos(\varphi-3\pi+2\pi x), \ldots$ $\ldots, \frac{3}{2}x-\frac{1}{2})$	$(\cos(\varphi-3\pi+2\pi x), \ldots$ $\ldots, \frac{3}{2}x-\frac{1}{2})$
$x=1$	$(\frac{1}{2}, 0, 2)$	● see below	$(r^{-1}(r(x)-\pi)\cos(\varphi-\pi), \ldots, 1)$.	$(\cos(\varphi-\pi), \sin(\varphi-\pi), 1)$

● $\frac{1}{2}\le x\le\frac{2}{3}$, $\varphi\in[-\pi,\pi]$, $m(\varphi) := r^{-1}(|\varphi|)+\frac{1}{2}(\frac{2}{3}-r^{-1}(|\varphi|))$:

	$\frac{1}{2}\le x\le r^{-1}(\lvert\varphi\rvert)$	$r^{-1}(\lvert\varphi\rvert)\le x\le m(\varphi)$	$m(\varphi)\le x\le\frac{2}{3}$
$\varphi\ne 0$ $\lvert\varphi\rvert\ne\pi$	$(\frac{1}{2}, 0, 2 - \frac{(x-\frac{1}{2})}{(r^{-1}(\lvert\varphi\rvert)-\frac{1}{2})}\cdot\frac{\lvert\varphi\rvert}{\pi})$	$(\frac{1}{2}, 0, 2-\frac{\lvert\varphi\rvert}{\pi} - \frac{(x-r^{-1}(\lvert\varphi\rvert))}{m(\varphi)-r^{-1}(\lvert\varphi\rvert)}(1-\frac{\lvert\varphi\rvert}{\pi}))$	$(\frac{1}{2}\cos((\frac{x-m(\varphi)}{\frac{2}{3}-m(\varphi)})(\varphi-\pi))$, $\frac{1}{2}\sin(\ldots), 1)$
$\varphi=0$		$(\frac{1}{2}, 0, 2-6(2x-1))$	$(\frac{1}{2}\cos((7-12x)\pi)$, $\frac{1}{2}\sin(\ldots), 1)$
$\lvert\varphi\rvert=\pi$	$(\frac{1}{2}, 0, 2-6(x-\frac{1}{2}))$		

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