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# A NOTE ON METRICALLY INWARD MAPPINGS

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Abstract: Fixed point theorems for multi-valued mappings, satisfying a certain inward condition are obtained.

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1. Introduction. Let (X,d) be a metric space, P(X) the class of all non-empty bounded closed subsets of X and D the Hausdorff metric on P(X) induced by d. Given a subset K of X, a mapping $T: K \longrightarrow P(X)$ is said to be (i) contractive on K if $D(T(x),T(y)) < d(x,y)$ for all x,y in K with $x \neq y$ and (ii) inward on K if for each x in K, there exists $v \in K$ such that $d(x,v) + d(v,T(x)) = d(x,T(x))$, where $v \neq x$ unless $d(x,T(x)) = 0$, where $d(x,T(x)) = \inf \{d(x,y) \mid y \in T(x)\}$. In case T is single-valued, the notion of "a contractive mapping" was first introduced by M. Edelstein in [3] and the notion of "an inward mapping" was called "a metrically inward mapping" in [2].

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The concept of inwardness for mappings defined on topological vector spaces was first studied by B.R. Halpern in his thesis [4]. Recently, many interesting results related to this concept have been obtained by F.E. Browder, Halpern-Bergman, K. Fan, Petryshyn-Fitzpatrick, W.A. Kirk, J. Caristi and by many others. See [2] and [5] for more detailed references.

2. Main results. W.A. Kirk pointed out ([2], Remarks) that Caristi's results Theorem (2.1)', Theorem 2.1 and hence also Theorem 2.2 can be proved by using a result of A. Brøndsted ([1], Theorem 2). For our purpose, we shall state a particular case of Brøndsted's result below.

Lemma 1. ([1], Theorem 2) Let $(M,d)$ be a complete metric space. If $\phi$ is a lower semi-continuous mapping from $M$ into $[0,\infty)$ then for each $x \in M$ there exists a point $u \in M$ such that $d(x,u) \leq \phi(x) - \phi(u)$ and $d(u,y) > \phi(u) - \phi(y)$ for all $y \in M$ with $y \neq u$.

We shall show that the above lemma can be used to generalize Caristi's results for multi-valued mappings:

Theorem 2: Let $(X,d)$ be a metric space and $K$ a non-empty complete subset of $X$. Suppose that $T: K \longrightarrow P(X)$ is inward on $K$ and is also a contraction:

$$D(T(x),T(y)) \leq k\, d(x,y), \text{ for all } x,y \in K$$

where $k \in [0,1)$ is a fixed constant. Then $T$ has a fixed point in $K$.

Proof. Define $\phi(x) = \frac{1}{1-k} d(x,T(x))$ for $x \in K$. Then

$\phi$ is continuous as T is a contraction. By Lemma 1, there exists $u \in K$ such that

(*) $\quad d(u,y) > \phi(u) - \phi(y)$, for all $y \in K$ with $y \neq u$.

We claim that $d(u,T(u)) = 0$. Suppose this were not true. Since T is inward on K, there exists $v \in K$ with $v \neq u$ such that

$$\begin{aligned} d(u,v) &= d(u,T(u)) - d(v,T(u)) \\ &\leq d(u,T(u)) - [d(v,T(v)) - D(T(v),T(u))] \\ &\leq d(u,T(u)) - d(v,T(v)) + k\, d(v,u) \end{aligned}$$

Thus $d(u,v) \leq \phi(u) - \phi(v)$, which contradicts (*). Therefore, $d(u,T(u)) = 0$ and hence $u \in T(u)$ since $T(u)$ is closed.

Another application of Lemma 1 gives us the following:

Theorem 3. Let $(M,d)$ be a complete metric space and f a mapping defined on M such that for each $x \in M$, $f(x)$ is a nonempty subset of M. Suppose that there exists a lower semicontinuous function $\phi : M \to [0,\infty)$ such that one of the following conditions holds:

(A) For each $x \in M$,
$D(x,f(x)) \leq \phi(x) - \phi(u)$, for some $u \in f(x)$.

(B) For each $x \in M$, $f(x)$ is compact and $d(x,f(x)) \leq$
$\leq \phi(x) - \phi(u)$, for all $u \in f(x)$.

Then there exists $u_0 \in M$ such that $u_0 \in f(u_0)$.

Next we shall show that if the set K in Theorem 2 is compact, then the condition that T being a contraction can be weakened to being "contractive".

Theorem 4. Let $(X,d)$ be a metric space and K a compact subset of X. Suppose $T: K \to P(X)$ is inward on K and is also contractive on K, then T has a fixed point in K.

**Proof.** Since $T$ is contractive on $K$ and $K$ is compact, there exists $u$ $K$ such that

$$d(u,T(u)) = \inf\{d(x,T(x)): x \in K\}.$$

We claim that $d(u,T(u)) = 0$. Suppose this were false. Since $T$ is inward on $K$, there exists $v \in K$ such that $v \neq u$ and $d(u,v) + d(v,T(u)) = d(u,T(u))$. Since $d(v,T(v)) \leq d(v,T(u)) + D(T(u),T(v))$ and since $T$ is contractive, one has $d(v,T(v)) < d(u,T(u))$, which contradicts the choice of $u$ in $K$. Thus $d(u,T(u)) = 0$. Hence $u \in T(u)$ since $T(u)$ is closed.

Finally, we remark that even when $T$ is single-valued, Theorem 2 (i.e. Theorem 2.2 in [2]) and Theorem 4 are incomparable in the sense that neither is more general than the other.

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