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MULTIVALUED GENERALIZED CONTRACTIONS AND FIXED POINT THEOREMS

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Abstract: We prove fixed point theorems for multivalued generalized contraction and contractive mappings in metrically convex metric spaces. Theorem 1 generalizes a fixed point theorem of Assad-Kirk for multivalued contraction mappings, Theorem 2 that of Assad for multivalued contractive mappings.

Key words: Multivalued generalized contraction (contractive) mapping, metrically convex metric space.

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1. Introduction. Recently fixed point theorems for multivalued contraction or contractive mappings were obtained by Nadler [9], Assad-Kirk [1] and Assad [2], etc. On the other hand, Kannan [5] initiated studies of certain type of mappings which have many similarities to contraction and nonexpansive mappings. His ideas were further studied and generalized by Reich [10], Čirič [3], Kannan [8], Hardy-Rogers [5], Goebel-Kirk-Shimi [4] and Wong [11, 12, 13], etc.

In this paper we shall give fixed point theorems for multivalued generalized contraction mappings and generalized contractive mappings. Theorem 1 is an extension of a theorem in Assad-Kirk [1]. Theorem 2 extends a fixed point theorem in Assad [2].

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2. Preliminaries. Let (X,d) be a metric space. For any $x \in X$ and $A \subset X$, we denote $d(x,A) = \inf\{d(x,y): y \in A\}$. It can easily be checked the following lemma.

Lemma 1. For any $x, y \in X$ and $A \subset X$, we have

$$|d(x,A) - d(y,A)| \leq d(x,y).$$

Let $\mathcal{CB}(X)$ denote the family of all nonempty closed bounded subsets of X and D be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X. The following lemmas are direct consequences of the definition of Hausdorff metric.

Lemma 2. If $A, B \in \mathcal{CB}(X)$ and $x \in A$, then for any positive number ε, there exists a $y \in B$ such that

$$d(x,y) \leq D(A,B) + \varepsilon .$$

Lemma 3. For any $x \in X$ and any $A, B \in \mathcal{CB}(X)$, it follows that

$$|d(x,A) - d(x,B)| \leq D(A,B).$$

(X,d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$, there exists an element $z \in X$, $x \neq z \neq y$, such that

$$d(x,z) + d(z,y) = d(x,y).$$

In Assad and Kirk [1] the following is noted.

Lemma 4. If K is a nonempty closed subset of a complete and metrically convex metric space (X,d), then for any $x \in K$, $y \notin K$, there exists a $z \in \partial K$ (the boundary of K) such that

$$d(x,z) + d(z,y) = d(x,y).$$

3. Generalized contraction mappings. Let K be a nonempty closed subset of a metric space (X,d) and T be a mapping of K into $\mathcal{CB}(X)$. T is said to be a generalized contraction mapping if there exist nonnegative real numbers α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ such that for any $x, y \in K$,

$$D(T(x),T(y)) \leq \alpha d(x,y) + \beta\{d(x,T(x)) + d(y,T(y))\} + \gamma\{d(x,T(y)) + d(y,T(x))\}.$$

If $\beta = \gamma = 0$, then T is called α-contraction.

The following theorem holds.

Theorem 1. Let (x,d) be a complete and metrically convex metric space, K a nonempty closed subset of X. Let T be a generalized contraction mapping of K into $\mathcal{CB}(X)$. If for any $x \in \partial K$, $T(x) \subset K$ and $\frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} < 1$, then there is a $z \in K$ such that $z \in T(z)$.

Proof. Denote $k = \frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2}$, then $0 \leq k < 1$. If $k = 0$, then the conclusion of Theorem 1 is obvious. So we may assume that $k > 0$. We choose sequences $\{x_n\}$ in K and $\{y_n\}$ in X in the following way. Let $x_0 \in \partial K$ and $x_1 = y_1 \in T(x_0)$. By Lemma 2, there exists a $y_2 \in T(x_1)$ such that

$$d(y_1,y_2) \leq D(T(x_0),T(x_1)) + \frac{1-\beta-\gamma}{1+\beta+\gamma} k.$$

If $y_2 \in K$, let $x_2 = y_2$. If $y_2 \notin K$, choose an element $x_2 \in K$ such that $d(x_1,x_2) + d(x_2,y_2) = d(x_1,y_2)$ using Lemma 4. By induction, we can obtain sequences $\{x_n\}$, $\{y_n\}$ such that for

$n = 1,2,\ldots,$

(1) $y_{n+1} \in T(x_n)$,

(2) $d(y_n, y_{n+1}) \leq D(T(x_{n-1}), T(x_n)) + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n$,

where

(3) $y_{n+1} = x_{n+1}$ if $y_{n+1} \in K$, or

(4) $d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1})$ if $y_{n+1} \notin K$.

We shall estimate the distance $d(x_n, x_{n+1})$ for $n \geq 2$.

There arise three cases.

(i) The case that $x_n = y_n$ and $x_{n+1} = y_{n+1}$. We have

$$d(x_n, x_{n+1}) = d(y_n, y_{n+1})$$
$$\leq D(T(x_{n-1}), T(x_n)) + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n$$
$$\leq \alpha d(x_{n-1}, x_n) + \beta\{d(x_{n-1}, T(x_{n-1})) + d(x_n, T(x_n))\}$$
$$+ \gamma\{d(x_{n-1}, T(x_n)) + d(x_n, T(x_{n-1}))\} + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n$$
$$\leq \alpha d(x_{n-1}, x_n) + \beta\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}$$
$$+ \gamma\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n,$$

hence

$$(1-\beta-\gamma) d(x_n, x_{n+1}) \leq (\alpha+\beta+\gamma) d(x_{n-1}, x_n) + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n$$

and

$$d(x_n, x_{n+1}) \leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} d(x_{n-1}, x_n) + \frac{k^n}{1+\beta+\gamma} .$$

(ii) The case that $x_n = y_n$ and $x_{n+1} \neq y_{n+1}$. By (4) we obtain that

$$d(x_n, x_{n+1}) \leq d(x_n, y_{n+1}) = d(y_n, y_{n+1}).$$

As in the case (i), we have

$$d(y_n,y_{n+1}) \leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\, d(x_{n-1},x_n) + \frac{k^n}{1+\beta+\gamma} ,$$

thus

$$d(x_n,x_{n+1}) \leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\, d(x_{n-1},x_n) + \frac{k^n}{1+\beta+\gamma} .$$

(iii) The case that $x_n \neq y_n$ and $x_{n+1} = y_{n+1}$. In this case $x_{n-1} = y_{n-1}$ holds. We have

$$d(x_n,x_{n+1}) \leq d(x_n,y_n) + d(y_n,x_{n+1}) = d(x_n,y_n) + d(y_n,y_{n+1}).$$

By (2) it follows that

$$d(y_n,y_{n+1}) \leq D(T(x_{n-1}),T(x_n)) + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n$$

$$\leq \alpha d(x_{n-1},x_n) + \beta\{d(x_{n-1},T(x_{n-1})) + d(x_n,T(x_n))\}$$

$$+ \gamma\{d(x_{n-1},T(x_n)) + d(x_n,T(x_{n-1}))\} + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n$$

$$\leq \alpha d(x_{n-1},x_n) + \beta\{d(x_{n-1},y_n) + d(x_n,x_{n+1})\}$$

$$+ \gamma\{d(x_{n-1},x_n) + d(x_n,x_{n+1}) + d(x_n,y_n)\} + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n.$$

Since $0 \leq \alpha < 1$ and $d(x_{n-1},x_n) + d(x_n,y_n) = d(x_{n-1},y_n)$, we obtain

$$d(x_n,x_{n+1}) \leq (1+\gamma)d(x_n,y_n) + (\alpha+\gamma)d(x_{n-1},x_n) +$$

$$+ \beta d(x_{n-1},y_n) + (\beta+\gamma)d(x_n,x_{n+1}) + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n$$

$$\leq (1+\gamma)d(x_{n-1},y_n) + \beta d(x_{n-1},y_n)$$

$$+ (\beta+\gamma)d(x_n,x_{n+1}) + \frac{1-\beta-\gamma}{1+\beta+\gamma} k^n,$$

and

$$d(x_n,x_{n+1}) \leq \frac{1+\beta+\gamma}{1-\beta-\gamma}\, d(x_{n-1},y_n) + \frac{k^n}{1+\beta+\gamma} .$$

As in the case (ii), we have

$$d(x_{n-1},y_n) \leq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\, d(x_{n-2},x_{n-1}) + \frac{k^{n-1}}{1+\beta+\gamma} .$$

Thus it follows that

$$d(x_n,x_{n+1}) \leq \frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} d(x_{n-2},x_{n-1}) + \frac{k^{n-1}}{1-\beta-\gamma} + \frac{k^n}{1+\beta+\gamma}.$$

The case that $x_n \neq y_n$ and $x_{n+1} \neq y_{n+1}$ does not occur. Since $\frac{\alpha+\beta+\gamma}{1-\beta-\gamma} \leq \frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2}$, for $n \geq 2$ we have

$$d(x_n,x_{n+1}) \leq \begin{cases} kd(x_{n-1},x_n) + \dfrac{k^n}{1-\beta-\gamma}, \text{ or} \\ kd(x_{n-2},x_{n-1}) + \dfrac{k^{n-1}+k^n}{1-\beta-\gamma}. \end{cases}$$

Put $\delta = k^{-\frac{1}{2}} \max(\|x_0 - x_1\|, \|x_1 - x_2\|)$, then by induction we can show that

$$d(x_n,x_{n+1}) \leq k^{\frac{n}{2}} \left(\delta + \frac{n}{1-\beta-\gamma}\right) \quad (n = 1,2,\ldots).$$

It follows that for any $m > n \geq 1$,

$$d(x_n,x_m) \leq \delta \sum_{i=m}^{m-1} (k^{\frac{1}{2}})^i + \frac{1}{1-\beta-\gamma} \sum_{i=m}^{m-1} i(k^{\frac{1}{2}})^i.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete and K is closed, $\{x_n\}$ converges to some point $z \in K$. By the way of choosing $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} = y_{n_i}$ $(i = 1,2,\ldots)$. Then we have

$$d(x_{n_i},T(z)) \leq D(T(x_{n_i-1}),T(z))$$

$$\leq \alpha d(x_{n_i-1},z) + \beta\{d(x_{n_i-1},T(x_{n_i-1})) + d(z,T(z))\}$$

$$+ \gamma\{d(x_{n_i-1},T(z)) + d(z,T(x_{n_i-1}))\}$$

$$\leq \alpha\{d(x_{n_i-1},x_{n_i}) + d(x_{n_i},z)\} + \beta\{d(x_{n_i-1},x_{n_i})$$

$$+ d(z,x_{n_i}) + d(x_{n_i},T(z))\} + \gamma\{d(x_{n_i-1},x_{n_i}) + d(x_{n_i},T(z)) + d(x_{n_i},z)\},$$

thus

$$(1-\beta-\gamma)d(x_{n_i},T(z)) \le (\alpha+\beta+\gamma)\{d(x_{n_i},z) + d(x_{n_i-1},x_{n_i})\}$$

and

$$d(x_{n_i},T(z)) \le \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\{d(x_{n_i},z) + d(x_{n_i-1},x_{n_i})\}.$$

Therefore, $d(x_{n_i},T(z)) \longrightarrow 0$ as $i \longrightarrow \infty$. By the inequality

$$d(z,T(z)) \le d(x_{n_i},z) + d(x_{n_i},T(z))$$

and the above result, it follows that $d(z,T(z)) = 0$. Since $T(z)$ is closed, this implies that $z \in T(z)$. q.e.d.

Since every Banach space is metrically convex, we have the following corollary for singlevalued mappings.

Corollary 1. Let E be a Banach space and K be a nonempty closed subset of E. Let f be a generalized contraction mapping of K into E. If $f(\partial K) \subset K$ and $\frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} < 1$, then there exists a (unique) fixed point of f in K.

3. Generalized contractive mappings. Let K be a nonempty closed subset of a metric space (X,d). Let T be a mapping of K into $\mathcal{CB}(X)$. T is said to be a generalized contractive mapping if there exist nonnegative real numbers α, β, γ such that for any $x, y \in K$ with $x \neq y$,

$$D(T(x),T(y)) < \alpha d(x,y) + \beta\{d(x,T(x)) + d(y,T(y))\} + \gamma\{d(x,T(y)) + d(y,T(x))\},$$

where $0 < \alpha + 2\beta + 2\gamma \leq 1$. If $\beta = \gamma = 0$ and $\alpha = 1$, then T is called contractive. T is said to be continuous at $x_0 \in K$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $D(T(x), T(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$. If T is continuous at each point of K, we say that T is continuous on K.

We shall give a fixed point theorem for continuous generalized contractive mappings.

Theorem 2. Let (X,d) be a complete and metrically convex metric space and K be a nonempty compact subset of X. Let T be a generalized contractive mapping of K into $\mathcal{CB}(X)$ and continuous on K. If for any $x \in \partial K$, $T(x) \subset K$ and $\frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} \leq 1$, then there exists an element $z \in K$ such that $z \in T(z)$.

Proof. Define a function g of K into R^+ (nonnegative real numbers) by $g(x) = d(x, T(x))$ $(x \in K)$, then by Lemma 1 and Lemma 3, we have

$$|g(x) - g(y)| \leq |d(x,T(x)) - d(y,T(x))| + |d(y,T(x)) - d(y,T(y))| \leq d(x,y) + D(T(x),T(y)).$$

Hence g is continuous and since K is compact, there exists a $z \in K$ such that $g(z) = \min\{g(x) : x \in K\}$. Suppose that $g(z) > 0$, then we obtain a contradiction. For each $n = 1, 2, \ldots$, there exists a $x_n \in T(z)$ for which

$$d(x_n, z) \leq g(z) + \frac{1}{n}.$$

If $x_n \in K$ for n sufficiently large, then some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to an $x_0 \in K$. We may assume that $x_0 \neq z$, then

$$g(x_0) = d(x_0,T(x_0)) \leq D(T(z),T(x_0))$$
$$< \alpha d(z,x_0) + \beta\{d(z,T(z)) + d(x_0,T(x_0))\}$$
$$+ \gamma\{d(z,T(x_0)) + d(x_0,T(z))\}$$
$$\leq \alpha g(z) + \beta\{g(z) + g(x_0)\} + \gamma\{g(z) + g(x_0)\}$$

and

$$(1 - \beta - \gamma)g(x_0) < (\alpha + \beta + \gamma)g(z).$$

Thus

$$g(x_0) < \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\, g(z) \leq g(z),$$

contradicting the minimality of $g(z)$. If there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \notin K$, then $z \notin \partial K$. For simplicity, we may assume that $x_n \notin K$, $n = 1,2,\ldots$. By Lemma 4, for each n there exists a $y_n \in \partial K$ for which $d(x_n,y_n) + d(y_n,z) = d(x_n,z)$. Since K is compact and $T(y_n) \subset K$, there exists $w_n \in T(y_n)$ such that $d(x_n,w_n) = d(x_n,T(y_n))$. We may also assume that $\{y_n\}$ converges to some $y_0 \in \partial K$. Let

$$8\varepsilon = \alpha d(y_0,z) + \beta\{d(y_0,T(y_0)) + d(z,T(z))\}$$
$$+ \gamma\{d(y_0,T(z)) + d(z,T(y_0))\} - D(T(y_0),T(z)),$$

then $\varepsilon > 0$, because $y_0 \neq z$. For this ε, there exists a positive integer N such that for any $n \geq N$

(5) $d(y_0,z) - d(y_n,z) < 2\varepsilon$,

(6) $g(y_0) - \varepsilon < g(y_n)$,

(7) $d(x_n,z) < g(z) + 2\varepsilon$, and

(8) $D(T(y_n),T(z)) < D(T(y_0),T(z)) + 2\varepsilon$

Then for any $n \geq N$, we have

$$g(y_0) - \varepsilon < g(y_n) = d(y_n,T(y_n))$$

$$\leq d(y_n,w_n) \leq d(y_n,x_n) + d(x_n,w_n) = d(x_n,y_n) + d(x_n,T(y_n))$$

$$\leq d(x_n,y_n) + D(T(z),T(y_n)) < d(x_n,y_n) + D(T(z),T(y_0)) + 2\varepsilon$$

$$= d(x_n,y_n) + \alpha\, d(y_0,z) + \beta\{d(y_0,T(y_0)) + d(z,T(z))\}$$

$$+ \gamma\{d(y_0,T(z)) + d(z,T(y_0))\} - 6\varepsilon$$

$$\leq d(x_n,y_n) + (\alpha + 2\gamma)d(y_0,z) + (\beta+\gamma)g(y_0) + (\beta+\gamma)g(z) -$$

$$- 6\varepsilon < (1+\beta+\gamma)g(z) + (\beta+\gamma)g(y_0) - 2\varepsilon \quad ,$$

hence

$$g(y_0) < \frac{1+\beta+\gamma}{1-\beta-\gamma}\, g(z) - \frac{\varepsilon}{1-\beta-\gamma}$$

Take a $u \in T(y_0)$ such that $d(y_0,T(y_0)) = d(y_0,u)$. Since $g(z) > 0$, $u \neq y_0$. Thus we obtain

$$g(u) = d(u,T(u)) \leq D(T(y_0),T(u))$$

$$< \alpha d(y_0,u) + \beta\{d(y_0,T(y_0)) + d(u,T(u))\}$$

$$+ \gamma\{d(y_0,T(u)) + d(u,T(y_0))\}$$

$$\leq (\alpha+\beta+\gamma)g(y_0) + (\beta+\gamma)g(u)$$

and

$$g(u) < \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\, g(y_0).$$

Therefore it follows that

$$g(u) < \frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2}\, g(z) - \frac{(\alpha+\beta+\gamma)\varepsilon}{(1-\beta-\gamma)^2}$$

$$\leq g(z) - \frac{(\alpha+\beta+\gamma)\varepsilon}{(1-\beta-\gamma)^2} .$$

This is a contradiction. Hence $g(z) = 0$ and since $T(z)$ is closed, we have $z \in T(z)$. q.e.d.

In Banach spaces, the following corollary holds.

Corollary 2. Let K be a nonempty compact subset of a Banach space E and f be a continuous generalized contractive mapping of K into E. If $f(\partial K) \subset K$ and $\frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} \leq 1$, then there exists a (unique) fixed point of f in K.

Remark. If for any $x \in K$, $T(x) \subset K$ in Theorem 1 (or Theorem 2), then the conditions that $k = \frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} < 1$ (or $k \leq 1$) and that X is metrically convex are unnecessary.

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