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A NEW METHOD FOR THE OBTAINING OF EIGENVALUES OF VARIATIONAL INEQUALITIES OF THE SPECIAL TYPE

(Preliminary communication)

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Abstract: Let A be a linear completely continuous operator in a Hilbert space H, K a cone in H, β a penalty operator corresponding to K. Under certain assumptions, there exist functions λ_ε, u_ε $(\varepsilon \in \langle 0,+\infty), \lambda_\varepsilon \in \mathbb{R}$, $u_\varepsilon \in H)$ starting in a given eigenvalue λ_0 and eigenvector u_0 of A, satisfying the equation $\lambda_\varepsilon u_\varepsilon - Au_\varepsilon + \varepsilon\beta u_\varepsilon = 0$ and converging to some eigenvalue λ_∞ and eigenvector u_∞ of the variational inequality.

Key words: Eigenvalues, variational inequality, operator of penalty.

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Let H be a real Hilbert space with the inner product $(.,.)$, K a closed convex cone in H, A a linear symmetric completely continuous operator of H into H. Suppose that A has only simple eigenvalues. We shall consider the following problem:

(I) $$u \in K,$$

(II) $$(\lambda u - Au, v - u) \geq 0 \text{ for all } v \in K,$$

where λ is a real parameter. A real number λ is said to be an eigenvalue of the variational inequality (I),(II), if there exists a nontrivial u satisfying (I),(II). In this case, u is said to be the corresponding eigenvector of the variational inequality (I),(II). It can be proved that if λ is an eigenvalue of (I),(II) with the corresponding eigenvector $u \in K^o$ *), then all the corresponding eigenvectors are on the half-line tu, $t > 0$ only. Especially, the following definition is reasonable.

Definition 1. We shall say that λ is a boundary eigenvalue and interior eigenvalue of the variational inequality (I),(II) if there exists the corresponding eigenvector $u \in \partial K$ and $u \in K^o$, respectively, of (I),(II). We shall say that λ is a boundary (with respect to K) eigenvalue and interior (with respect to K) eigenvalue of the operator A if there exists the corresponding eigenvector $u \in \partial K$ and $u \in K^o$, respectively, of the operator A.

Let us consider a nonlinear completely continuous operator β of H into H (a penalty operator corresponding to K) satisfying the following assumptions:

(1) $\beta u = 0$ if and only if $u \in K$;

(2) $(\beta u - \beta v, u - v) \geq 0$ for all $u, v \in H$;

(3) β is differentiable on H - K in the sense of Fréchet;

(4) if $u \in K^o$, $v \notin K$, then $(\beta v, u) \neq 0$;

(5) if $\varepsilon_n > 0$, $u_n \in H$ $(n = 1,2,\ldots)$ and the sequence $\{\varepsilon_n \beta u_n\}$ is bounded, then $\{\varepsilon_n \beta u_n\}$ contains a strongly convergent subsequence;

(6) for each fixed $u \in H - K$, $\varepsilon > 0$, a linear operator $\beta'(u)$ is symmetric and $A - \varepsilon \beta'(u)$ has only simple eigen-

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*) We denote by ∂K and K^o the boundary and interior of K, respectively.

values.

Moreover, we shall consider the following assumption about the connection between the solution of the nonlinear equation with the penalty and the corresponding linearized equation ($R > 0, \Lambda_2 > \Lambda_1 > 0$ are given numbers):

If $\lambda \in \langle \Lambda_1, \Lambda_2 \rangle$, $\varepsilon \in \langle 0, R \rangle$, $u \in H - K$, $v \in H$, $\|u\| = \|v\| = 1$,

(NL) (i) $\lambda u - Au + \varepsilon \beta u = 0$,

(ii) $\lambda v - Av + \varepsilon \beta'(u)(v) = \mu u$ for some real μ,

then $(u,v) \neq 0$.

<u>Theorem 1</u>. Let $\lambda^{(1)}$ be interior eigenvalue of A, $\lambda^{(o)}$ an eigenvalue of A corresponding to the eigenvector $u^{(o)} \notin K$, $\|u^{(o)}\| = 1$, $0 < \lambda^{(1)} < \lambda^{(o)}$. Suppose that there is no boundary eigenvalue of A in the interval $\langle \lambda^{(1)}, \lambda^{(o)} \rangle$. Let the assumptions (1 - 6) be fulfilled and let (NL) hold with $\Lambda_1 = \lambda^{(1)}$, $\Lambda_2 = \lambda^{(o)}$, $R = +\infty$. Then there exist differentiable functions λ_ε, u_ε on $\langle 0, +\infty)$ such that $\lambda_o = \lambda^{(o)}$, $u_o = u^{(o)}$, λ_ε is decreasing and the following conditions hold for all $\varepsilon \geq 0$:

(a) $\|u_\varepsilon\| = 1$, $u_\varepsilon \notin K$, $\lambda^{(1)} < \lambda_\varepsilon < \lambda^{(o)}$,

(b) $\lambda_\varepsilon u - Au_\varepsilon + \varepsilon \beta u = 0$.

Moreover, $\lambda_\varepsilon \longrightarrow \lambda_\infty^{(o)}$ (as $\varepsilon \longrightarrow +\infty$) and $u_{\varepsilon_n} \rightharpoonup u_\infty^{(o)}$ **) (for some sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$, $\varepsilon_n \longrightarrow +\infty$), where $\lambda^{(1)} < \lambda_\infty^{(o)} < \lambda^{(o)}$, $u_\infty^{(o)} \in \partial K$, $\lambda_\infty^{(o)}$ is a boundary eigenvalue and $u_\infty^{(o)}$ is the corresponding eigenvector of (I),

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**) $\longrightarrow$ and $\rightharpoonup$ denotes the strong and weak convergence, respectively.

(II). If $\{\varepsilon_n\}$ is an arbitrary sequence such that $\varepsilon_n >$ > 0, $\varepsilon_n \longrightarrow +\infty$, $u_{\varepsilon_n} \longrightarrow u_\infty$ ***), then u_∞ is also the eigenvector of (I),(II) corresponding to $\lambda_\infty^{(o)}$ and $u_\infty \in \partial K$, $u_{\varepsilon_n} \longrightarrow u_\infty$.

For a trivial illustration, we can consider the following example. (More complicated examples will be discussed in [1], § 5.) Consider the Sobolev space $H = \overset{o}{W}{}_2^1(\langle 0,1 \rangle)$ with the inner product

$$(u,v) = \int_0^1 u'v'dx,$$

and the cone $K = \{u \in H;\ u(x_i) \geq 0,\ i = 1,\dots,n\}$, where $x_i \in$ $\in (0,1)$, $i = 1,\dots,n$, are given. Define the operators A and β_α ($\alpha \in \langle 0,1)$) by

$$(Au,v) = \int_0^1 u\ v\ dx \text{ for all } u,\ v \in H,$$

$$(\beta_\alpha u,v) = -\sum_{i=1}^{n} |u(x_i)|^\alpha u^-(x_i)v(x_i) \text{ for all } u,\ v \in H.$$

If $n = 1$ (i.e. K is a half-space), then all assumptions of Theorem 1 can be verified for the operator $\beta = \beta_0$. (The condition (NL) holds with $\Lambda_1 = 0$, $\Lambda_2 = +\infty$, $R = +\infty$.) For $n > 1$ the assumption (3) is not fulfilled for $\beta = \beta_0$. In this case, the assumptions of more complicated Theorem 2 formulated below are satisfied for $\beta^{(n)} = \beta_{\frac{1}{n}}$ and $\beta =$ $= \beta_0$ (see [1], § 5).

Let us consider a penalty operator β which does not satisfy the condition (3). We shall suppose that there exists a sequence $\beta^{(n)}$ of completely continuous operators

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***) See p. 207 Footnote

such that

(7) if $\{u_n\}$ is bounded, then $\{\beta^{(n)}u_n\}$ contains a strongly convergent subsequence; if $u_n \rightharpoonup u$, then $\beta^{(n)}u_n \longrightarrow \beta u$.

<u>Theorem 2</u>. Let $\lambda^{(1)}$, $\lambda^{(2)}$ be interior eigenvalues of A, $\lambda^{(o)}$ an eigenvalue of A corresponding to the eigenvector $u^{(o)} \notin K$, $\|u^{(o)}\| = 1$, $0 < \lambda^{(1)} < \lambda^{(o)} < \lambda^{(2)}$. Suppose that there is no boundary eigenvalue of A in the interval $\langle \lambda^{(1)}, \lambda^{(2)} \rangle$. Consider that β fulfils (1),(2), (4),(5),(6) and $\beta^{(n)}$ for each fixed n fulfil (1),(3),(4), (5),(6). Suppose that for each $R > 0$ there exists n_0 such that (NL) is valid with R and $\Lambda_1 = \lambda^{(1)}$, $\Lambda_2 = \lambda^{(2)}$ for each $\beta^{(n)}$, $n > n_0$. Let the condition (7) be satisfied. Then for each $\varepsilon \geq 0$ there exists at least one couple λ_ε, u_ε satisfying the condition (b) and

(a′) $\|u_\varepsilon\| = 1$, $u_\varepsilon \notin K$, $\lambda^{(1)} < \lambda_\varepsilon < \lambda^{(2)}$.

Moreover, there exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n > 0$ $\varepsilon_n \longrightarrow +\infty$, $\lambda_{\varepsilon_n} \longrightarrow \lambda_\infty^{(o)}$, $u_{\varepsilon_n} \longrightarrow u_\infty^{(o)}$, where $\lambda_\infty^{(o)} \in$ $\in (\lambda^{(1)}, \lambda^{(2)})$, $u_\infty^{(o)} \in \partial K$, $\lambda_\infty^{(o)}$ is a boundary eigenvalue and $u_\infty^{(o)}$ is the corresponding eigenvector of (I),(II). If $\{\varepsilon_n\}$ is arbitrary such that $\varepsilon_n > 0$, $\varepsilon_n \longrightarrow +\infty$, $\lambda_{\varepsilon_n} \longrightarrow \lambda_\infty$, $u_{\varepsilon_n} \rightharpoonup u_\infty$, then λ_∞ is also the boundary eigenvalue and u_∞ the corresponding eigenvector of (I),(II), $\lambda_\infty \in (\lambda^{(1)}, \lambda^{(2)})$, $u_\infty \in \partial K$, $u_{\varepsilon_n} \longrightarrow u_\infty$.

If A has infinitely many of interior eigenvalues then our theory ensures the existence of infinitely many of boundary eigenvalues of (I),(II). The obtained eigenvectors are not simultaneously eigenvectors of A.

The proof of the abstract result is based on the abstract implicit function theorem (see [1], § 3).

R e f e r e n c e

[1] M. KUČERA: A new method for the obtaining eigenvalues of variational inequalities. Branches of eigenvalues of the equation with the penalty. To appear.

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