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TERNARY RINGS ASSOCIATED TO TRANSLATION PLANE

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Abstract: It is well known that an affine plane is a translation plane if and only if there exists a quasifield coordinatizing it. Simple condition for planary ternary ring with zero coordinatizing a translation plane is deduced by Klucký and Marková in [4]. We shall define a J-ternary ring or JTR to be a PTR that $\exists 0 \in S$ such that
$T(a,0,c) = T(a,b,c)$ implies $T(a,0,y) = T(a,b,y)$ $\forall y \in S$
$T(0,a,c) = T(b,a,c)$ implies $T(0,a,y) = T(b,a,y)$ $\forall y \in S$.
In [5] Martin defines an intermediate ternary ring (ITR). Strucurally, the JTR lie between the PTR and ITR. The purpose of this note is to deduce a necessary and sufficient condition that a given JTR coordinatizes a translation plane. This generalizes the main results of [4] and [5].

Key words: Planar ternary ring, translation plane, intermediate ternary ring, generalized Cartesian group.

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A coordinatization of a projective plane: We shall give a coordinatization to a projective plane of order n. Let S be any set of cardinality n. Let ∞ be any element which is not in S and let $0 \in S$. We pick one point L and one line ℓ joining through L in the plane. For any $M \in \ell$ denote by $\widetilde{M}$ the set of all lines containing M. Let $m \mapsto (m)$ be a bijection of $S \cup \{\infty\}$ onto ℓ such that $[\infty] = \ell$. Let $x \mapsto [x]$ be a bijection of $S \cup \{\infty\}$ onto $\widetilde{L}$ such that $[\infty] = \ell$. Let $y \mapsto (0,y)$ be a bijection of S onto $[0] \setminus \{L\}$. We denote by $A \sqcup B$ ($a \sqcap b$) the line joining two distinct points A,B (the common point of two distinct lines). Let α_1, α_2: $S \longrightarrow S$ be two mappings. Then to every point P off $\mathcal{L}$ we assign coordinates (x,y) if and only if $P = [x] \sqcap ((\alpha_1(x)) \sqcup (0,y))$. We shall now dualize the above construction in the following sense. Let $c \mapsto [0,c]$ be a bijection of S onto $(\tilde{0}) \setminus \{\mathcal{L}\}$. Then to every line p off $\tilde{L}$ we assign coordinates [m,c] if and only if $p = (m) \sqcup ([\alpha_2(m)] \sqcap$ $\sqcap [0,c])$.

Planar ternary rings:

Definition 1: Let S be a set containing two different elements at least and let ternary operation T be given on it. An ordered pair (S,T) will be called a planar ternary ring or PTR if it holds:

(1) $\forall a,b,c \in S \ \exists! x \in S \qquad T(a,b,x) = c$

(2) $\forall a,b,c,d \in S; x \in S \qquad T(x,a,b) = T(x,c,d)$

(3) $\forall a,b,c,d \in S;\ a \neq c \ \exists (x,y) \in S^2 \quad T(a,x,y) = b,\ T(c,x,y) = d$

An intermediate ternary ring on ITR (see [5],p.1187) is a PTR (S,T) such that (I_1) and (I_2) holds.

(I_1) $T(m,a,y) = T(m,b,y) = c,\ a \neq b$ implies $T(m,x,y) = c$ $\forall x \in S$

(I_2) $T(a,x,y) = T(b,x,y) = c,\ a \neq b$ implies $T(m,x,y) = c$ $\forall m \in S$

A J-ternary ring or JTR is a PTR (S,T) such that there exists $0 \in S$ where

(J_1) $T(m,0,a) = T(m,x,a)$ implies $T(m,0,y) = T(m,x,y)$ $\forall y \in S$

(J_2) $T(0,x,a) = T(m,x,a)$ implies $T(0,x,y) = T(m,x,y)$ $\forall y \in S$

Let (S,T) be a PTR. Then (S,T) defines a projective plane $\pi(S,T)$ as follows.

Points: $(x,y),(m),(\infty)$; $m,x,y\in S$, ∞ not in S

Lines: $[m,c] := \{(x,y) \mid x,y\in S,\ T(m,x,y)=c\}$

$[x] := \{(x,y) \mid y\in S\}$

$[\infty] := \{(\infty)\} \cup \{(m) \mid m\in S\}$

In [2],[3](p. 114-115),[5](p. 1186) there was shown that $\pi(S,T)$ is a projective plane. Thus a solution in (3) is unique.

Proposition 1: *Let π be a projective plane. Then there exists a JTR (S,T) such that $\pi(S,T)$ is isomorphic to π.*

Proof: Let the projective plane π be coordinatized as above by elements from a set S. Define a ternary operation by $T(m,x,y) = c$ if and only if (x,y) is on $[m,c]$. Then it is obvious that the (S,T) is a JTR. One has only to check (1),(2),(3),(J_1),(J_2) in turn.

Remark: Let (S,T) be a JTR. Then there are mappings α_1, α_2: $S \longrightarrow S$ such that $\forall x,y\in S$ $T(\alpha_1(x),0,y) = T(\alpha_1(x),x,y)$

$\forall m,y\in S$ $T(0,\alpha_2(m),y) = T(m,\alpha_2(m),y)$

and such that for every point (x,y) and every line $[m,c]$ in $\pi(S,T)$ is $(x,y) = [x] \cap ((\alpha_1(x)) \cup (0,y))$

$[m,c] = (m) \cup ((\alpha_2(m)) \cap [0,c])$

Proposition 2: *Let (S,T) be an ITR. Then (S,T) is a JTR.*

Proof: The proposition is a direct consequence of Theorem 6 in [5], p. 1188.

Vertically transitive planes: (S,t) is said to be the dual ternary system of PTR (S,T) if $c = T(m,x,t(x,m,c))$

$\forall m,c,x\in S$ or equivalently $y = t(x,m,T(m,x,y))$ $\forall m,x,y\in S$.

<u>Proposition 3</u>: *The dual of a JTR is a JTR.*

<u>Proof</u>: The proof is straightforward.

In the following we shall denote by j_a^1 the solution of the equation $t(x,0,0) = t(x,a,a)$ for each $a\in S\setminus\{0\}$ and by j_a^2 the solution of the equation $T(x,0,0) = T(x,a,a)$ for each $a\in S\setminus\{0\}$; additionally we define $j_o^1 = j_o^2 = 0$. Thus for each $a\in S$ is $t(j_a^1,0,0) = t(j_a^1,a,a)$ and $T(j_a^2,0,0) = T(j_a^2,a,a)$. Now let us introduce in S two binary operations $+_1, +_2$ by virtue of

$$a +_1 b := T(a,j_a^1,t(j_a^1,0,b))$$
$$a +_2 b := t(a,j_a^2,T(j_a^2,0,b)) \qquad \forall a,b\in S$$

<u>Remark</u>: It can be easily verified that

(4) $0 +_1 a = a +_1 0 = 0 +_2 a = a +_2 0 = a \qquad \forall a\in S$

(5) $\forall a,b\in S\ \exists! x\in S \qquad a +_1 x = b$

$\forall a,b\in S\ \exists! y\in S \qquad a +_2 y = b$

<u>Definition 2</u>: Let (S,T) be a PTR. The projective plane $\pi(S,T)$ is said to be a vertically transitive plane (by [4], p. 620) if for each $x,y,z\in S$ there exists a translation τ of the affine plane $(S^2,\{[m,c] \mid m,c\in S\} \cup \{[x] \mid x\in S\})$ such that $(x,y)^\tau = (x,z)$.

Let (S,T) be a JTR and (S,t) its dual. By (1) $\varphi_1: y \mapsto T(0,0,y)$, $\varphi_2: c \mapsto t(0,0,c)$ are bijective mappings and $\varphi_1\varphi_2 = \varphi_2\varphi_1 = id$.

<u>Proposition 4</u>: *Let (S,T) be a JTR. Then the projective plane $\pi(S,T)$ is a vertically transitive plane if and only if*

(6) $\forall m,c,x,y \in S\ (T\ m,x,y +_2 c) = T(m,x,y) +_1 (0^{\rho_2} +_2 c)^{\rho_1}$

Proof. I. Suppose first that (S,T) (6) holds. We shall see that $(S,+_2)$ is a loop. By (4),(5) it is sufficient to show that $\forall u,c \in S\ \exists! v \in S\ v +_2 c = u$.

Let $a +_2 c = b +_2 c$ and let $m,x \in S$ such that $x \neq 0$, $T(m,0,a) = T(m,x,b)$. Then

$T(m,0,a +_2 c) = T(m,0,a) +_1 (0^{\rho_2} +_2 c)^{\rho_1} =$

$= T(m,x,b) +_1 (0^{\rho_2} +_2 c)^{\rho_1} = T(m,x,b +_2 c)$ and by (J_1)

$T(m,0,a) = T(m,x,a) = T(m,x,b)$ hence $a = b$. Now let $u \in S$. Choose $m,x,y \in S$ such that $x \neq 0$

$T(m,0,u) = T(m,x,y +_2 c)$ and denote

$(0,v) := [m,T(m,x,y)] \sqcap [0]$. Then there is $T(m,0,v) =$

$= T(m,x,y)$, $T(m,0,u) = T(m,x,y +_2 c) = T(m,x,y) +_1 (0^{\rho_2} +_2 c)^{\rho_1} =$

$= T(m,0,v) +_1 (0^{\rho_2} +_2 c)^{\rho_1} = T(m,0,v +_2 c)$ from here $v +_2 c =$

$= u$.

Thus, the map $\tau_c : S^2 \longrightarrow S^2$ defined by

$(x,y)^{\tau_c} := (x,y +_2 c)$ is a translation. Since $(0,0)^{\tau_c} =$

$= (0,c)$, the $\pi(S,T)$ is a vertically transitive plane.

II. Let $\pi(S,T)$ be a vertically transitive plane. Then for each $a \in S$ there is a translation τ_a mapping $(0,0)$ into $(0,a)$. Then $(y,y)^{\tau_a} = (y,y +_2 a)$ for each $y \in S$ hence $(0,y)^{\tau_a} =$

$= (0,y +_2 a)$ for each $y \in S$ and $(x,y)^{\tau_a} = (x,y +_2 a)$ for each $x,y \in S$. It is obvious that $[0,0]^{\tau_a} = [0,(0^{\rho_2} +_2 a)^{\rho_1}]$

this implies $[m,c]^{\tau_a} = [m,c +_1 (0^{\rho_2} +_2 a)^{\rho_1}]$. Hence,

$(x,y) \in [m,T(m,x,y)]$ for each $m,x,y \in S$ from here $(x,y)^{\tau_a} \in$

$\in [m,T(m,x,y)]^{\tau_a}$ then $(x,y +_2 a) \in [m,T(m,x,y) +_1 (0^{\rho_2} +_2 a)^{\rho_1}]$

consequently $T(m,x,y +_2 a) = T(m,x,y) +_1 (0^{\rho_2} +_2 a)^{\rho_1}$

for each $m,x,y,a \in S$.

Corollary 4.1: *Let* (S,T) *be a JTR and let* $\pi(S,T)$ *be a vertically transitive plane. Then* $(S,+_1)$, $(S,+_2)$ *are groups and* $(S,+_1)$ *is isomorphic to* $(S,+_2)$.

Proof: Consider translations $\varphi:(0,0)\mapsto(0,a)$, $\sigma:(0,0)\mapsto(0,b)$, $\tau:(0,0)\mapsto(0,c)$. Then $(0,(a+_2 b)+_2 c) = (0,0)^{(\varphi\sigma)\tau} = (0,0)^{\varphi(\sigma\tau)} = (0,a+_2(b+_2 c))$.

The second result follows from (6). In particular, for every $a,b\in S$ $(a+_2 b)^{\varphi_1} = T(0,0,a+_2 b) = T(0,0,a)+_1(0^{\varphi_2}+_2 b)^{\varphi_1} = a^{\varphi_1}+_1(0^{\varphi_2}+_2 b)^{\varphi_1}$. Since for each $y,a,b\in S$ $y+_2(a+_2 b)=(y+_2 a)+_2 b$, we have $y^{\varphi_1}+_1(0^{\varphi_2}+_2(a+_1 b))^{\varphi_1} = (y+_2(a+_2 b))^{\varphi_1} = (y+_2 a)^{\varphi_1}+_1(0^{\varphi_2}+_2 b)^{\varphi_1} = (y^{\varphi_1}+_1(0^{\varphi_2}+_2 a)^{\varphi_1})+_1 +_1(0^{\varphi_2}+_2 b)^{\varphi_1}$.

Setting $y = 0^{\varphi_2}$, we have

$(0^{\varphi_2}+_2(a+_1 b))^{\varphi_1} = (0^{\varphi_2}+_2 a)^{\varphi_1}+_1(0^{\varphi_2}+_2 b)^{\varphi_1}$.

Remark: The group of all translations of a vertically transitive plane $\pi(S,T)$ is Abelian if and only if $(S,+_1)$ is commutative.

Now let us introduce two binary operations $\cdot_1$, $\cdot_2$ by virtue of

$T(m,x,0) = m\cdot_1 x \qquad \forall\, m,x\in S$

$t(x,m,0) = x\cdot_2 m \qquad \forall\, m,x\in S$

Corollary 4.2: *Let* (S,T) *be a JTR and let* $\pi(S,T)$ *be a vertically transitive plane. Then*

(7) $\forall\, m,x,y\in S \quad T(m,x,y) = m\cdot_1 x+_1(0^{\varphi_2}+_2 y)^{\varphi_1}$

$t(x,m,y) = x\cdot_2 m+_2(0^{\varphi_1}+_1 y)^{\varphi_2}$

Proof: Let as set $y = 0$ in (6). Then
$T(m,x,c) = m \cdot_1 x +_1 (0^{\varphi_2} +_2 c)^{\varphi_1}$ for each $m,x,c \in S$.

Proposition 5: *Let (S,T) be a JTR. The projective plane $\pi(S,T)$ is a vertically transitive plane if and only if*

(8) $(S,+_1)$, $(S,+_2)$ *are groups*

(9) *there exists an isomorphism* $\varphi : (S,+_2) \longrightarrow (S,+_1)$ *such that* $\forall m,x,y \in S$ $T(m,x,y) = m \cdot_1 x +_1 y^{\varphi}$.

Proof: I. Let (8),(9) hold for (S,T). Then for each $m,x,y,c \in S$ $T(m,x,y +_2 c) = m \cdot_1 x +_1 (y +_2 c)^{\varphi} = m \cdot_1 x +_1 (y^{\varphi} +_1 c^{\varphi}) = (m \cdot_1 x +_1 y^{\varphi}) +_1 c^{\varphi} = T(m,x,y) +_1 c^{\varphi}$.
Setting $m = x = 0$, $y = 0^{\varphi_2}$, we have $(0^{\varphi_2} +_2 c)^{\varphi_1} = 0 +_1 c^{\varphi}$ thus $c^{\varphi} = (0^{\varphi_2} +_2 c)^{\varphi_1}$ for each $c \in S$ therefore $T(m,x,y +_2 c) = T(m,x,y) +_1 (0^{\varphi_2} +_2 c)^{\varphi_1}$ for each $m,x,y,c \in S$.

II. The second part follows immediately from Corollary 4.1 and Corollary 4.2.

Corollary 5.1: *Let (S,T) be a JTR such that $T(0,0,y) = y$ for each $y \in S$. Then the projective plane $\pi(S,T)$ is a vertically transitive plane if and only if*

(i) $(S,+_1)$ is a group

(ii) $\forall m,x,y \in S$ $T(m,x,y) = m \cdot_1 x +_1 y$

Proof: I. $\forall m,x,y,c \in S$ $T(m,x,y +_1 c) = m \cdot_1 x +_1 (y +_1 c) = (m \cdot_1 x +_1 y) +_1 c = T(m,x,y) +_1 c$.
Hence $\pi(S,T)$ is a vertically transitive plane.

II. If $\pi(S,T)$ is a vertically transitive plane, then by Proposition 5 $(S,+_1)$ is a group and there exists an isomorphism $\varphi : (S,+_2) \rightarrow (S,+_1)$ such that $T(m,x,y) = m \cdot_1 x +_1 y^{\varphi}$ for each $m,x,y \in S$. This yields then $y = T(0,0,y) = 0 +_1 y^{\varphi} = y^{\varphi}$ for each $y \in S$ hence $T(m,x,y) = m \cdot_1 x +_1 y$ for each $m,x,y \in S$.

Corollary 5.2: *Let* (S,T) *be a JTR and* (S,t) *its dual. Let* π(S,T) *be a vertically transitive plane, then there exists an isomorphism* $\varphi:(S,+_2)\longrightarrow(S,+_1)$ *such that* $\forall m,x,y\in S$ $T(m,x,y)=m\cdot_1 x+_1 y^{\varphi}$,

$$t(x,m,y)=x\cdot_2 m+_2 y^{\varphi^{-1}},\quad m\cdot_1 x+_1(x\cdot_2 m)^{\varphi}=0$$

Proof: Since it holds $T(m,x,t(x,m,0))=0$ for each $m,x\in S$, we have $m\cdot_1 x+_1(x\cdot_2 m)^{\varphi}=0$. Since it holds $T(m,x,t(x,m,y))=y$ for each $m,x,y\in S$, we obtain $m\cdot_1 x+_1(t(x,m,y))^{\varphi}=y$ thus $(t(x,m,y))^{\varphi}=-_1 m\cdot_1 x+_1 y=(x\cdot_2 m)^{\varphi}+_1 y$ from what you say $t(x,m,y)=x\cdot_2 m+_2 y^{\varphi^{-1}}$.

Definition 3: Let S be a set +,$\cdot$ two binary operations on S. (S,+,$\cdot$) will be called a generalized Cartesian group (see [4],p. 620) if S has two distinct elements at least and if it holds:

(10) (S,+) is a group

(11) $\forall a,b,c\in S;\ a\neq b\ \exists!\,x\in S \qquad -xa+xb=c$

(12) $\forall a,b,c\in S;\ a\neq b\ \exists\,x\in S \qquad ax-bx=c$

Propoposition 6: *Let* $\mathbb{C}:=(S,+,\cdot)$ *be a generalized Cartesian group and let* $\varphi: S\longrightarrow S$ *be a bijection such that* $0^{\varphi}=0$. *If we define* $T(\mathbb{C},\varphi)$, $(m,x,y)=m\cdot x+y^{\varphi}$ *for each* $m,x,y\in S$ *then* (S,T($\mathbb{C},\varphi$)) *is a JTR and* π(S,T($\mathbb{C},\varphi$)) *is a vertically transitive plane.*

Proof: The proof is straightforward. One has only to check (1),(2),(3),(J_1),(J_2),(8),(9) in turn.

Proposition 5 and Proposition 6 now imply the next

Theorem 1: *Let* (S,T) *be a JTR. Then the projective plane* π(S,T) *is a vertically transitive plane if and only if*

(*i*) $\mathbb{C}:=(S,+_1,\cdot_1)$ *is a generalized Cartesian group* *(ii) there exists a bijection* $\varphi: S \longrightarrow S$ *such that* $O^{\varphi} = O$, $T = T(\mathbb{C}, \varphi)$.

<u>Translation planes</u>: First we give some general remarks. Let us investigate a projective plane $\pi = (P,L)$. Let us distinguish a line ℓ. Then by an affine plane $\pi(\ell)$ we shall as usual mean the restriction of π to the incidence structure $(P \setminus \ell, \{m \setminus (m \cap \ell) \mid m \in L \setminus \{\ell\}\})$. The points from $P \setminus \ell$ will be called proper, the points of ℓ improper or directions. A projective plane $\pi = (P,L)$ is said to be an ℓ-transitive plane if the group of all translations of $\pi(\ell)$ transitively operates on the set of all points of $\pi(\ell)$. Let u, v be affine lines of $\pi(\ell)$ with different directions, then the projective plane π is a ℓ-transitive plane if and only if the group of all translations of $\pi(\ell)$ transitively operates on the lines u, v.

<u>Proposition 7:</u> *Let* $\mathbb{C} = (S,+,\cdot)$ *be a generalized Cartesian group and* $\varphi: S \longrightarrow S$ *a bijection such that* $O^{\varphi} = O$. *Then the projective plane* $\pi(S, T(\mathbb{C}, \varphi))$ *is a* $[\infty]$-*transitive plane if and only if*

(13) $\forall x, a \in S \; \exists x' \in S \; \forall m \in S$

$mx' - Ox' = ma - Oa + OO - mO + mx - Ox$

<u>Proof</u>: I. It suffices to prove that the group of all translations transitively operates on proper points of the line $[O,O]$ In this case it suffices to show that for each line $[a]$ there exists a translation τ such that $[O]^{\tau} = [a]$. Define a mapping $\tau_a : (x,y) \longmapsto (x', (-Ox' + Ox + y^{\varphi})^{\varphi^{-1}}$ with $x' \in S$ uniquely determined by (13) (see (12)). Clearly τ_a is bijective. Further it is obvious that the image of

the line $[x]$ is the line $[x']$. Let us consider a line $[m,c]$. If $(x,y) \in [m,c]$, then $T(\mathbb{C},\varphi)\quad(m,x,y) =$ $mx + y^{\varphi} = c$. Hence it is

$T(\mathbb{C},\varphi)\quad(m,x',(-0x'+0x+y^{\varphi})^{\varphi^{-1}}) =$

$= (mx' - 0x') + 0x + y^{\varphi} = (ma - 0a + 00 - m0 + mx - 0x) +$

$+ 0x + y^{\varphi} = (ma - 0a + 00 - m0) + c$

or equivalently $(x',(-0x'+0x+y^{\varphi})^{\varphi^{-1}}) \in$

$\in [m, ma - 0a + 00 - m0 + c]$. If $x = x'$ for some $x \in S$, then necessarily $a = 0$ therefore $\tau_a = id$. This implies τ_a is a translation. Setting $x=0$ in (13), we obtain $m0' - 00' = ma - 0a$ for each $m \in S$ then $0' = a$ hence $[0]^{\tau_a} = [a]$ and consequently $\pi(S,T(\mathbb{C},\varphi))$ is a $[\infty]$-transitive plane.

II, Conversely, suppose that $\pi(S,T(\mathbb{C},\varphi))$ is a $[\infty]$-transitive plane. First of all, evidently for $a=0$ $mx - 0x =$ $= ma - 0a + 00 - m0 + mx - 0x$ for each $m,x \in S$. Thus suppose $a \neq 0$. For $x=0$ we have $ma - 0a = ma - 0a + 00 - m0 + mx - 0x$ for each $m \in S$. Thus suppose $x \neq 0$. Now choose any element $k \in S \setminus \{0\}$. By (12) there is x' such that

$kx' - 0x' = ka - 0a + 00 - k0 + kx - 0x$

Further let τ_a be a translation for which $(0,0)^{\tau_a} =$

$= (a, (-0a + 00 + 0^{\varphi})^{\varphi^{-1}})$. Then

$(0,0),(x,(-kx + k0 + 0^{\varphi})^{\varphi^{-1}}) \in [k, k0 + 0^{\varphi}]$,

$(0,0),(a,(-0a + 00 + 0^{\varphi})^{\varphi^{-1}}) \in [0, 00 + 0^{\varphi}]$,

$(a(-0a + 00 + 0^{\varphi})^{\varphi^{-1}}),(x',(-0x' + 0x - kx + k0 + 0^{\varphi})^{\varphi^{-1}}) \in$

$\in [k, ka - 0a + 00 + 0^{\varphi}]$

$(x,(-kx + k0 + 0^{\varphi})^{\varphi^{-1}}),(x',(-0x' + 0x - kx + k0 + 0^{\varphi})^{\varphi^{-1}}) \in$

$\in [0, 0x - kx + k0 + 0^{\varphi}]$.

Thus, $(x,(-kx + k0 + 0^{\varphi})^{\varphi^{-1}})^{\tau_a} = (x',(-0x' + 0x - kx +$ $+ k0 + 0^{\varphi})^{\varphi^{-1}})$ hence $[x]^{\tau_a} = [x']$. For $m=0$ is

$0 = mx' - Ox' = ma - Oa + OO - mO + mx - Ox$.

Thus let be $m \in S \setminus \{O\}$. Then

$(x, (-mx + mO + O^{\varphi})^{\varphi-1}) \in [O, Ox - mx + mO + O^{\varphi}]$,

$(x', (-Ox' + Ox - mx + mO + O^{\varphi})^{\varphi-1}) \in [O, Ox - mx + mO + O^{\varphi}] \cap$

$\cap [x']$. Thus, $(x, (-mO + mO + O^{\varphi})^{\varphi-1})^{\tau_a} =$

$= (x', (-Ox' + Ox - mx + mO + O^{\varphi})^{\varphi-1})$.

But $T(\mathbb{C}, \varphi)(m, O, O) = mO + O^{\varphi} =$

$= T(\mathbb{C}, \varphi)(m, x, (-mx + mO + O^{\varphi})^{\varphi-1})$ and then it follows

necessarily $T(\mathbb{C}, \varphi)(m, a, (-Oa + OO + O^{\varphi})^{\varphi-1}) =$

$= T(\mathbb{C}, \varphi)(m, x', (-Ox' + Ox - mx + mO + O^{\varphi})^{\varphi-1})$, hence

$ma - Oa + OO + O^{\varphi} = mx' - Ox' + Ox - mx + mO + O^{\varphi}$ consequently $mx' - Ox' = ma - Oa + OO - mO + mx - Ox$

Thus Proposition 7 is proved.

Corollary 7.1.: *Let* $(S, +, \cdot)$ *be a generalized Cartesian group such that the condition (13) holds. Then the group* $(S, +)$ *is Abelian.*

Proof: The proof of the preceding corollary depends on the obvious fact that the group of all translations of a $[\infty]$-transitive plane is Abelian.

Proposition 8: *Let* $\mathbb{C} = (S, +, \cdot)$ *be a generalized Cartesian group such that there exists* $e \in S$ *where for each* $x \in S$ $e \cdot x = e \cdot O$. *Further let* $\varphi : S \longrightarrow S$ *be a bijection such that* $O^{\varphi} = O$. *Then the projective plane* $\pi(S, T(\mathbb{C}, \varphi))$ *is a* $[\infty]$*-transitive plane if and only if*

(14) $\forall x, a \in S \; \exists x' \in S \; \forall m \in S \; mx' - mx = ma - mO$

Proof: I. First we note that by (12) for every $a \in S \setminus \{e\}$ and for every $b \in S$ there exists exactly one $x \in S$ such that $ax - ex = b - eO$ it holds if and only if $ax - eO = b - eO$,

$ax = b$. This implies that for each $a \in S\setminus\{e\}$ and for each $b \in S$ there exists exactly one $x \in S$ such that $ax = b$. Define a mapping $\tau_a: (x,y) \longmapsto (x',y)$ with $x' \in S$ uniquely determined by (14). Clearly τ_a is a bijective. Further it is obvious that the image of the line $[x]$ is the line $[x']$. Let us consider a line $[m,c]$. If $(x,y) \in [m,c]$, then
$T(\mathbb{C},\varphi)(m,x,y) = mx + y^{\varphi} = c$. Hence it is
$T(\mathbb{C},\varphi)(m,x',y) = mx' + y^{\varphi} = ma - mO + mx + y^{\varphi} = ma - mO + c$
or equivalently $(x',y) \in [m, ma - mO + c]$. If $x' = x$ for some $x \in S$ then by (14) $a = O$, $\tau_a = id$. This implies τ_a is a translation. Setting $x = O$ in (14), we have $mO' = ma$ for each $m \in S$ hence $[O]^{\tau_a} = [a]$ and consequently $\pi(S,T(\mathbb{C},\varphi))$ is a $[\infty]$-transitive plane.

II. Let $\pi(S,T(\mathbb{C},\varphi))$ be a $[\infty]$-transitive plane. Setting $m = e$ in (13), we obtain $ex' - Ox' = ea - Oa + OO - eO + ex - Ox$ then $-Ox' = -Oa + OO - Ox$ hence $mx' - Ox' = mx' - Oa + OO - Ox = ma - Oa + OO - mO + mx - Ox$ for each $m \in S$ and by Corollary 7.1 $mx' = ma - mO + mx$ therefore $mx' - mx = ma - mO$.

Theorem 1 and Proposition 7 now imply

<u>Theorem 2</u>: *Let (S,T) be a JTR. Then the projective plane $\pi(S,T)$ is a $[\infty]$-transitive plane if and only if*

(*i*) $\mathbb{C} := (S,+_1,\cdot_1)$ *is a generalized Cartesian group*

(*ii*) *there exists a bijection $\varphi: S \longrightarrow S$ such that* $O^{\varphi} = O$, $T = T(\mathbb{C},\varphi)$

(*iii*) $\forall\, x,a \in S\ \exists\, x' \in S\ \forall\, m \in S$

$$m\cdot_1 x' -_1 O\cdot_1 x' = m\cdot_1 a -_1 O\cdot_1 a +_1 O\cdot_1 O -_1 m\cdot_1 O +_1 m\cdot_1 x -_1 O\cdot_1 x$$

R e f e r e n c e s

[1] HALL M.: Projective planes, Trans. Amer. Math. Soc. 54 (1943), 229-277.

[2] HAVEL V.: A general coordinatization principle for projective planes with comparison of Hall and Hughes frames and with examples of generalized oval frames (to appear in Czech. Math. Journal).

[3] HUGHES D.R. and PIPER F.C.: Projective planes, Springer Verlag New York-Heidelberg-Berlin 1973.

[4] KLUCKÝ D. and MARKOVÁ L.: Ternary rings with zero associated to translation planes, Czech. Math. Journal 23 (98) (1973), 617-628.

[5] MARTIN G.E.: Projective planes and isotopic ternary rings, Amer. Math. Monthly 74 (1967) II, 1185-1195.

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