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CONVERGENCE OF CONDITIONAL EXPECTATIONS

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Abstract: A simple lemma in which uniform integrability together with convergence in distribution implies convergence in probability is presented. The result provides a generalization to that of D. Gilat (1971) and Štěpán (1971).

Key words and phrases: Bayes estimator, uniform integrability, convergence in distribution, convergence in probability.

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The purpose of this note is to present a result in which uniform integrability together with convergence in distribution implies convergence in probability. The result, which provides a generalization to that of D. Gilat (1971), is designed to show that the sequence of Bayes estimators of a real valued function is consistent with respect to L_r-convergence ($r \geq 1$) if and only if it is consistent with respect to convergence in distribution. Our main result is

Lemma. Let $\{X_n\}$, $\{Y_n\}$ be sequences of integrable random variables such that X_n, Y_n are defined on a probability

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1) Part of this work was performed while the author was visiting the Mathematical Institute of the University of Aarhus, Danmark.

space $(\Omega_n, \mathcal{A}_n, P_n)$. Suppose that $E[X_n \mid \mathcal{E}_n] \le Y_n$ [1] where $\mathcal{E}_n \subset \mathcal{A}_n$, $n \ge 1$, are σ-algebras and assume the sequences $\{X_n^-\}$, $\{Y_n^-\}$ to be uniformly integrable. If X_n and Y_n have the same limiting distribution then $X_n - Y_n \xrightarrow{P} 0$ [2].

Moreover, if

(1) $E[X_n \mid \mathcal{E}_n] = Y_n$, $n \ge 1$ and $|X_n|^r$ is uniformly integrable for some $r \ge 1$,

so is $|Y_n|^r$; hence this lemma implies $E|X_n - Y_n|^r \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof of Lemma. First [3] consider the stronger set of assumptions (1) putting there $r = 1$. Fix a positive integer k and define Φ by

$$\Phi(t) = \begin{cases} t^2 & 0 \le t \le k \\ 2kt - k^2 & t > k \\ \Phi(-t) & t < 0. \end{cases}$$

Φ is continuous, linear for $|t| \ge k$. Hence the uniform integrability argument (Loeve (1963), page 183) applies to conclude from our assumptions that $E\,\Phi(X_n) - E\,\Phi(Y_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

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1) Equalities and inequalities between random variables are meant in the almost sure sense.

2) We write $X_n - Y_n \xrightarrow{P} 0$ and mean that $X_n - Y_n \longrightarrow 0$ in probability as $n \longrightarrow \infty$, i.e. $P_n[\,|X_n - Y_n| \ge \varepsilon\,] \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\varepsilon > 0$.

3) The method employed in the first part of this proof is due to the referee of the present note. The author's original proof was much more complicated.

Further define Ψ by

$$\Psi(x,t) = 2xt - x^2 \qquad |x| \leq k,\ t \in R^1$$
$$= 2kt - k^2 \qquad x > k,\ t \in R^1$$
$$= -2kt - k^2 \qquad x < -k,\ t \in R^1;$$

i.e. $t \longrightarrow \Psi(x,t)$ is the unique linear function which is $\leq \Phi$ and equal Φ at the point x. Moreover, for any given $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\Phi(t) - \Psi(x,t) \geq \delta \quad \text{if } |x - t| \geq \varepsilon \quad \text{and } |x| \leq k - 1.$$

Since

$$E[\Psi(Y_n, X_n) \mid \mathcal{E}_n] = \Phi(Y_n), \qquad n \geq 1$$

we arrive at

$$[E\Phi(X_n) - E\Phi(Y_n)] \geq \delta P_n[|X_n - Y_n| \geq \varepsilon, |Y_n| \leq k - 1] \longrightarrow 0$$

as $n \longrightarrow \infty$. Letting $k \longrightarrow \infty$ it is easy to argue from the tightness of the sequence $\{Y_n\}$ that $X_n - Y_n \xrightarrow{\pi} 0$.

Finally, consider $\{X_n\}$, $\{Y_n\}$ satisfying the hypotheses of Lemma. Take $c > 0$ and put

$$\Delta(t) = t \qquad t \leq c$$
$$= c \qquad t > c.$$

The conditional form of Jensen's inequality (Loeve (1963), page 348) provides the argument for the inequality

$$Z_n = E[\Delta(X_n) \mid \mathcal{E}_n] \leq \Delta(Y_n) \qquad n \geq 1$$

since Δ is continuous concave and nondecreasing. From the uniform integrability of $\{X_n^-\}$, $\{Y_n^-\}$ it follows that $E\Delta(X_n) - E\Delta(Y_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Consequently

$$Z_n - \Delta(Y_n) \xrightarrow{\pi} 0. \tag{2}$$

To prove that $\Delta(X_n) - \Delta(Y_n) \xrightarrow{\pi} 0$, which is obviously sufficient for our purposes, we simply apply the proven part of this lemma to the sequences $\{Z_n\}$, $\{\Delta(X_n)\}$ ($\Delta(X_n)$ is uniformly integrable) and combine the result with (2).

The following example shows that our lemma is not necessarily true if its uniform integrability assumptions are not satisfied. Let the $(\Omega, \mathcal{A}, P)$ be the closed unit interval with Lebesgue measure. Denote by I_A the indicator of a set A and put for $n \geq 1$

$$A_n = [0, \tfrac{1}{2n}),\ B_n = [\tfrac{1}{2n}, \tfrac{1}{2}),\ C_n = [\tfrac{1}{2}, 1 - \tfrac{1}{2n}),$$

$$D_n = [1 - \tfrac{1}{2n}, 1],$$

$$X_n = -n.I_{A_n} + I_{C_n} + n.I_{D_n},\quad \mathcal{E}_n = \sigma(A_n \cup C_n, B_n \cup D_n),$$

$$Y_n = E[X_n \mid \mathcal{E}_n].$$

Simple computations show that the sequences X_n, Y_n have the same limiting distribution but the sequence $X_n - Y_n$ fails to converge in probability to zero.

A pair of random variables is said to be fair (subfair) if $E[X \mid Y] = Y$ ($E[X \mid Y] \leq Y$). D. Gilat (1971) introduced this concept and proved that if (Y,X) is a subfair pair of integrable random variables then Y and X have the same distribution if and only if $X = Y$. Obviously, our Lemma provides a generalization to this result.

As a corollary we obtain the following comparison of L_r-convergence and convergence in distribution:

<u>Corollary 1</u> (J. Štěpán (1971)). Consider random variables $X, X_1, X_2, \ldots$ whose r-th ($r \geq 1$) absolute moments are finite such that $X_n \longrightarrow X$ in distribution as $n \longrightarrow \infty$. Then $E|X_n - X|^r \longrightarrow 0$ if and only if $E|E[X \mid X_n] - X_n|^r \longrightarrow 0$ as $n \longrightarrow \infty$.

Finally, consider a parameter-space Θ which is endowed with a priori probability distribution μ defined on a σ-algebra $\mathcal{B}$ of its subsets and have a sequence of statistical problems where the n-th term of the sequence consists of a measurable sample space $(Z_n, \mathcal{E}_n)$ and a family of probability measures $\{P_{n\theta}, \theta \in \Theta\}$ which are defined on $\mathcal{E}_n$. Moreover, suppose that the mapping $P_{n\theta}(E): \Theta \longrightarrow R^1$ is measurable for $E \in \mathcal{E}_n$.

The objects under consideration determine a sequence of probability spaces $(\Omega_n, \mathcal{A}_n, P_n)$, $n \geq 1$ where

$$\Omega_n = Z_n \times \Theta \ , \quad \mathcal{A}_n = \mathcal{E}_n \times \mathcal{B} \quad \text{and}$$

$$P_n(E \times B) = \int_B P_{n\theta}(E)\, \mu(d\theta) \qquad E \in \mathcal{E}_n,\ B \in \mathcal{B} \ .$$

Considering $f: \Theta \longrightarrow R^1$, a measurable and integrable function, the sequence of conditional expectations

$$b_n(f) = E_{P_n}[f \mid \mathcal{E}_n] \qquad n \geq 1$$

is called the Bayes estimator of f. (By $\mathcal{E}_n$ we mean the natural extension of the original σ-algebra such that $\mathcal{E}_n \subset \mathcal{A}_n$.)

Thus, we may apply the assertion of Lemma to get

Corollary 2. Consider $r \geq 1$ and a function $f: \theta \longrightarrow R^1$ such that $|f|^r$ is integrable. Then the Bayes estimator converges to f in distribution if and only if

$$\lim_{n \to \infty} E_{P_n} | b_n(f) - f|^r = 0.$$

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