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THE CATEGORIES OF FREE METABELIAN GROUPS AND LIE ALGEBRAS

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Abstract: Homomorphisms of free metabelian A_qA-groups, $q \geq 0$, and free metabelian Lie algebras over a commutative associative unital ground ring k are studied. It is proved that the group of automorphisms of a free metabelian Lie algebra L of rank 2, identical on L/L' is isomorphic to the additive group of the polynomial group $k[X,Y]$. Further; If $f: L_1 \longrightarrow L_2$ is an epimorphism of free A_qA-groups or metabelian Lie algebras over a ring $k = k_0[X_1,\dots,X_r, X_{r+1}^{\pm 1},\dots,X_s^{\pm 1}]$, where k_0 is a Dedekind ring, $rkL_1 = n$, $rkL_2 = d$, then L_1 possesses a free generating set $z_1,\dots,z_n$ such that $f(z_1),\dots,f(z_d)$ is a free generating set for L_2 and $z_{d+1},\dots,z_n$ generate Ker f as a normal subgroup or an ideal.

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The present paper concerns homomorphisms of free metabelian A_qA-groups, $q \geq 0$, and free metabelian Lie algebras over a commutative associative unital ground ring k. In § 2 we show that the group of automorphisms of a free metabelian Lie algebra L of rank 2, identical on L/L' (IA-automorphisms in terms of [1]) is isomorphic to the additive group of the polynomial group $k[X,Y]$. For comparison the similar group for a free metabelian A^2-group consists of inner automorphisms (see [1]).

In § 3 and 4 we show that if $f\colon L_1 \twoheadrightarrow L_2$ is an epimorphism of free A_qA-groups or metabelian Lie algebras over a ring $k = k_0[X_1,\dots,X_r, X_{r+1}^{\pm 1},\dots,X_s^{\pm 1}]$, where k_0 is a Dedekind ring, $rkL_1 = n$, $rkL_2 = d$, then L_1 possesses a free generating set $z_1,\dots,z_n$ such that $f(z_1),\dots,f(z_d)$ is a free generating set for L_2 and $z_{d+1},\dots,z_n$ generate Ker f as a normal subgroup or an ideal. In particular, let P be a retract of a free metabelian A_qA-group or Lie k-algebra L with a projection $f\colon L \twoheadrightarrow P$, k as above with k_0 a principal ideal ring. Then by [2] P is free and L possesses a free generating set $z_1,\dots$ $\dots,z_n$ such that $f(z_i) \equiv z_i$ mod Kerf in addition to the properties mentioned above.

A consideration of metabelian Lie algebras is motivated by the following reason. If k is a field, chark = 0, then any proper subvariety of metabelian Lie algebras is nilpotent (see [3]). Moreover, this variety is semisimple,[4]. By [5] if L is a free nilpotent algebra over a field with a retract P then P is a free factor of L. A trivial example in § 3 shows that this does not hold for metabelian Lie algebras.

It is worthy of mention that the similar results for absolutely free linear algebras were exhibited in [6].

§ 1. <u>Homomorphisms of free metabelian Lie algebras</u>. First we need a representation of free metabelian Lie algebras of finite rank n. Let $K = k[X_1,\dots,X_n]$ be a polynomial ring with the augmentation ideal $\mathfrak{m} = (X_1,\dots,X_n)$ and M a free

K-module with the base $e_1,\dots,e_n$. Define an epimorphism of K-modules

$$\mathcal{l}: M \longrightarrow \mathfrak{M}, \quad \mathcal{l}(e_i) = X_i.$$

Then M can be regarded as a k-algebra with the multiplication

(1) $$ab = \mathcal{l}(b)a - \mathcal{l}(a)b, \quad a, b \in M.$$

A direct calculation shows that M is a metabelian Lie algebra. Put

$$L = \{a \in M \mid \mathcal{l}(a) = \sum_{i=1}^{n} \alpha_i X_i, \quad \alpha_i \in k\}$$

Theorem 1. L is a subalgebra in M and a free metabelian Lie algebra with the base $e_1,\dots,e_n$.

The proof under assumption that k is a field was given in [7]. But this restriction on k was not used in the proof and is not necessary.

Corollary. $L' = \operatorname{Ker} \mathcal{l}$.

Proof. If $a, b \in L$, then by (1) $\mathcal{l}(ab) = 0$. Conversely, if

$$a = \sum \alpha_i e_i \mod L', \quad \alpha_i \in k,$$

and $\mathcal{l}(a) = 0$, then $\mathcal{l}(a) = \sum \alpha_i X_i$ implies $\alpha_1 = \dots = \alpha_n = 0$ and $a \in L'$.

Consider now two free metabelian Lie algebras L_1, L_2 over k with the bases $e_1,\dots,e_n$ and $u_1,\dots,u_d$. Let

$$K_1 = k[X_1,\dots,X_n], \quad K_2 = k[Y_1,\dots,Y_d]$$

and $M_i, K_i, \mathfrak{M}_i, \mathcal{l}_i$ be associated with L_i, $i = 1,2$, by Theorem 1. Given any homomorphism $\varphi : K_1 \longrightarrow K_2$ of k-algebras such that

(2) $$\varphi(X_i) = \sum \varphi_{ij} Y_j \ , \quad \varphi_{ij} \in k,$$

consider a φ-semilinear homomorphism $h\colon M_1 \longrightarrow M_2$ of modules making commutative the following diagram

(2′) $$\begin{tikzcd} M_1 \arrow[r, "\ell_1"'] \arrow[d, "h"'] & \mathcal{M}_1 \arrow[d, "\varphi"] \\ M_2 \arrow[r, "\ell_2"'] & \mathcal{M}_2 \end{tikzcd}$$

<u>Proposition 1</u>. h is a homomorphism of Lie algebras, defined by (1), and $h(L_1) \subseteq L_2$.

<u>Proof</u>. If $a, b \in M_1$ then by (1) and (2′)

$$h(ab) = h(\ell_1(b)a - \ell_1(a)b) = \varphi(\ell_1(b))h(a) - \varphi(\ell_1(a))h(b) =$$
$$= \ell_2(h(b))h(a) - \ell_2(h(a))h(b) = h(a)h(b).$$

Also by (2) and Theorem 1 we have $h(L_1) \subseteq L_2$.

Now we show that every homomorphism $f\colon L_1 \longrightarrow L_2$ can be extended to a unique semilinear homomorphism (h, φ) with the properties (2),(2′). In order to do this define $\varphi : K_1 \longrightarrow K_2$ as $\varphi(X_i) = \ell_2(f(e_i))$. Note that by (2′) and Theorem 1 this is the unique way of defining φ . Define also $h\colon M_1 \longrightarrow M_2$ by $h(e_i) = f(e_i)$.

<u>Proposition 2</u>. If $a \in L_1$, then $f(a) = h(a)$.

<u>Proof</u>. The case $a = e_i$ follows from definition. If $f(a_j) = h(a_j)$, then $f(\sum \alpha_j a_j) = h(\sum \alpha_j a_j)$. Now let $f(a) = h(a)$, $f(b) = h(b)$. In this case

$f(ab) = f(a)f(b) = \ell_2(f(b))f(a) - \ell_2(f(a))f(b) =$
$= \ell_2(h(b))h(a) - \ell_2(h(a))h(b) = h(a)h(b) = h(ab)$

by Proposition 1.

Thus we have proved

<u>Theorem 2</u>. Each semilinear map (h, φ) with (2),(2′) defines a homomorphism $f\colon L_1 \longrightarrow L_2$ of free metabelian Lie algebras and conversely every homomorphism $f\colon L_1 \longrightarrow L_2$ of Lie algebras has a unique representation by a semilinear morphism of modules.

By uniqueness the correspondence between morphisms of Lie algebras and semilinear morphisms is functorial. Starting from now we identify homomorphism $f\colon L_1 \longrightarrow L_2$ with its semilinear representation (h, φ).

§ 2. <u>Automorphisms of free metabelian Lie algebras</u>. In this part we consider the case $L_1 = L_2 = L$ and $f = (h, \varphi) \in$ $\in$ Aut L. By the corollary from Theorem 1 an automorphism f is identical on L/L' iff $\varphi = 1$. Let G be a group of all these automorphisms (IA-automorphisms in terms of [1]). It is clear that $G \lhd$ Aut L and by [5] Aut L is a semidirect product of GL(n,k) and G. By (2′) $f = (h,1) \in G$ iff h is an automorphism of M as K-module, that is $h \in GL(n,K)$, and $\ell(a) =$ $= \ell(f(a))$ for all $a \in M$. If $e_1, \dots, e_n$ is a base of M, $\ell(e_i) =$ $= X_i$, then $h = (h_{ij})$, where $h(e_i) = \sum_{j=1}^{n} e_j h_{ji}$ and

$$(3) \qquad X_i = \ell(e_i) = \ell(h(e_i)) = \sum_{j=1}^{n} X_j h_{ji}$$

This implies $h_{ij} = \delta_{ij} + g_{ij}$, where $\sum_{i=1}^{n} X_i g_{ij} = 0$, $j = 1, \dots$ $\dots, n$. Hence,

$$h = E + T \in SL(n,K),\ T = (g_{ij})$$

In particular for $n = 2$ we have

$$T = \begin{pmatrix} X_2t_1 & X_2t_2 \\ -X_1t_1 & -X_1t_2 \end{pmatrix} \quad t_1, t_2 \in k[X_1, X_2]$$

and

$1 = \det(E + T) = (1 + X_2t_1)(1 - X_1t_2) + X_1X_2t_1t_2 = 1 +$
$+ X_2t_1 - X_1t_2$, that is $t_1 = X_1t, t_2 = X_2t$. Hence,

$$T = \begin{pmatrix} X_1X_2t & X_2^2t \\ -X_1^2t & -X_1X_2t \end{pmatrix} = T(t)$$

Note that $T(t)T(t') = 0$ and thus for $E + T(t)$, $E + T(t') \in G$ we have

$$(E + T(t))(E + T(t')) = E + T(t + t')$$

Thus, we have proved

<u>Theorem 3</u>. If L is a free metabelian Lie algebra of rank 2, then Aut L is a semidirect product of $GL(2,k)$ and a group G of IA-automorphisms isomorphic to the additive group of $k[X_1, X_2]$.

§ 3. <u>Epimorphisms of free metabelian Lie algebras</u>. In this part we assume that for all s, r the group $GL(s, k[X_1, \dots ..., X_r])$ acts transitively on unimodular rows (see [8]). This is equivalent to the following fact: if $R = k[X_1, \dots, X_s]$ and M is R-module such that $R^s \simeq M \oplus R^p$ then $M \simeq R^{s-p}$. The fundamental result of [8] shows that this condition is satisfied when $k = k_0[Y_1, \dots, Y_n, Z_1^{\pm 1}, \dots, Z_r^{\pm 1}]$, where k_0 is a Dedekind ring.

Let $L_i, K_i, M_i, \mathfrak{M}_i, \ell_i$, $i = 1,2$, be as in § 1 and $f: L_1 \longrightarrow L_2$ an epimorphism, $f = (h, \varphi)$, $\mathrm{rk} L_1 = n$, $\mathrm{rk} L_2 = d$. Since L_2 is projective it can be regarded as a retract of L_1, that is L_2 is a subalgebra in L_1 and there is a projection $f: L_1 \longrightarrow L_2$ identical on L_2, i.e. $f^2 = f$. By (2), Theorem 2 and the remark made after this theorem φ is an idempotent endomorphism of $K_1 = k[X_1,\dots,X_n]$, where $\varphi(X_i) = \sum \varphi_{ij}X_j$, $\varphi_{ij} \in k$. Thus φ is an idempotent endomorphism of a free k-module $kX_1 + \dots + kX_n \simeq k^n$ and $\mathrm{Im}\,\varphi \simeq k^d$ since L_2 is free. By the remark made above $\mathrm{Ker}\,\varphi \simeq k^{n-d}$ and thus

$$K = k[X_1,\dots,X_n] = k[Y_1,\dots,Y_n]$$

for some $Y_1,\dots,Y_n$, where

$$\varphi(Y_i) = \begin{cases} Y_i, & i = 1,\dots,d; \\ 0, & i = d+1,\dots,n. \end{cases} \tag{4}$$

Let $\alpha = (\alpha_{ij}) \in GL(n,k) \subseteq \mathrm{Aut}\, K$ and $Y_i = \alpha(X_i) = \sum_j \alpha_{ij}X_j$, $i = 1,\dots,n$. Then the map g, $g(e_i) = \sum_j \alpha_{ij}e_j$ defines an α-semilinear map (g,α) for

$$\ell_1(g(e_i)) = \sum \alpha_{ij}X_j = Y_i = \alpha(X_i) = \alpha(\ell_1(e_i)).$$

Thus without loss of generality we can suppose from the very beginning that in (4)

$$\varphi(X_i) = \begin{cases} X_i, & i = 1,\dots,d; \\ 0, & i = d+1,\dots,n. \end{cases} \tag{4'}$$

Let $\mathfrak{M}_2$ be the augmentation ideal $(X_1,\dots,X_d) \lhd k[X_1,\dots,X_d]$, $\mathrm{Im}\,h = M_2$, and $J = (X_{d+1},\dots,X_n) \lhd k[X_1,\dots,X_n]$, $f = (h,\varphi)$, where φ from (4′). Then the diagram (2′) looks as

$$\begin{tikzcd} M_1 \arrow[r, "\ell_1"] \arrow[d, "h"'] & \mathfrak{m}_1 \arrow[d, "\varphi"] \\ M_2 \arrow[r, "\ell_2"] & \mathfrak{m}_2 \end{tikzcd} \tag{5}$$

Note that by (4') $JM_1 \subseteq \operatorname{Ker} h$ and hence (5) induces a commutative diagram

$$\begin{tikzcd} M_1' = M_1/JM_1 \arrow[r, "\ell_1'"] \arrow[d, "h'"'] & \mathfrak{m}_1/J\mathfrak{m}_1 = \mathfrak{m}_2 \arrow[d, "1"] \\ M_2 \arrow[r, "\ell_2"] & \mathfrak{m}_2 \end{tikzcd} \tag{5'}$$

Now M_1' is a free $K_2 = k[X_1,\dots,X_d]$-module with the base $e_i' = e_i + JM_1$, $1 \leq i \leq n$, and by (5') h' is an epimorphism of free K_2-modules. As we have already noticed Ker h is a free K_2-module of rank $n - d$. Now we can identify M_1' with $\sum_{i=1}^{m} K_2 e_i \subseteq M_1$. Thus we choose in M_1 a new base $w_1,\dots,w_n \in \sum_{i=1}^{m} K_2 e_i$ such that $h(w_1),\dots,h(w_d)$ is a base for M_2 and $w_{d+1},\dots,w_n \in \operatorname{Ker} h$. Moreover, $\operatorname{Ker} \varphi = J$. Since $X_i + \mathfrak{m}_1^2$, $i = 1,\dots,n$, is a base of a free k-module $\mathfrak{m}_1/\mathfrak{m}_1^2$ by (4') we can also assume that

$$H = \begin{pmatrix} \ell_1(w_1) \\ \vdots \\ \vdots \\ \ell_n(w_n) \end{pmatrix} \equiv X \bmod J, \text{ where } X = \begin{pmatrix} X_1 \\ \vdots \\ \vdots \\ X_n \end{pmatrix}$$

for we can always suppose that $\ell_2(h(w_i)) = X_i$, $i = 1,\dots,d$, and $w_j \in \operatorname{Ker} h$ implies $\ell_1(w_j) \in J$. Thus H is $\mathfrak{m}_1$-modular (see

[2],[7]).

Consider now a subgroup $D \subseteq GL(n,K_1)$ generated by $GL(n,K_1,J)$ (see [9]) and all matrices

$$\begin{pmatrix} A & U \\ 0 & B \end{pmatrix}, \quad A \in GL(d,K_1), \quad B \in GL(n-d,K_1).$$

Proposition 3. There exists $C \in D$ such that $CH = X$.

The proof in a more general situation will be given in Proposition 4.

Since $w_{d+1},\dots,w_n \in \operatorname{Ker} h$, $JM_1 \subseteq \operatorname{Ker} h$ by Proposition 3 for a new base $u_i = Cw_i$, $i = 1,\dots,n$ in M_1 we have

$$\ell_1(u_i) = X_i, \; i = 1,\dots,n; \; u_j \in \operatorname{Ker} h, \; j = d+1,\dots,n,$$

and $h(u_1),\dots,h(u_d)$ is a base for M_2. Thus we have proved

Theorem 4. Let k be a ring such that $GL(s,k[X_1,\dots$ $\dots,X_r])$ acts transitively on sets of unimodular columns for all s, r. If $f: L_1 \longrightarrow L_2$ is an epimorphism of free metabelian Lie algebras over k, $\operatorname{rk} L_1 = n$, $\operatorname{rk} L_2 = d$, then L_1 possesses a free base $u_1,\dots,u_n$ such that $f(u_1),\dots,f(u_d)$ is a base for L_2 and $u_{d+1},\dots,u_n$ generate $\operatorname{Ker} f$ as an ideal. In particular, the theorem holds for $k = k_0[Y_1,\dots,Y_c,Z_1^{\pm 1},\dots,Z_p^{\pm 1}]$, where k_0 is a Dedekind ring (see [8]).

Corollary. Let k be as above with k_0 a principal ideal ring, L a free metabelian Lie algebra over k, $\operatorname{rk} L = n$, and P a retract of L, $\operatorname{rk} P = d$ (see [2],[7]). If $f: L \longrightarrow P$ is a projection, then L possesses a free base $u_1,\dots,u_n$ with the properties of Theorem 4 such that in addition $f(u_i) \equiv u_i \bmod \operatorname{Ker} f$, $i = 1,\dots,d$.

Proof. By [2] P is free and $f(a) - a \in \operatorname{Ker} f$ for all $a \in L$

since $f^2 = f$.

Now we need to prove Proposition 3. Following [2] consider a more general situation: let $A_0 \subset A_1 \subset \dots \subset A_n \subset \dots$ be a chain of commutative rings, $1 \in A_0$ and for all i

1) A_i is a retract of A_{i+1} with kernel (X_{i+1});

2) each X_i is not a zero divizor;

3) if $\mathfrak{M}_i = (X_1,\dots,X_i) \lhd A_i$, then $\mathfrak{M}_i/\mathfrak{M}_i^2$ is a free A_0-module of rank i;

4) $GL(t,A_i)$ acts transitively on sets of unimodular columns for all $t \geq i$.

__Proposition 4.__ Let H be a column of length $t \geq n$, that is an element of a free A_n-module A_n^t, $J = (X_{d+1},\dots,X_n) \lhd A_n$ and

$$H \equiv X = \begin{pmatrix} X_1 \\ \cdot \\ X_n \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \mod J$$

If H is $\mathfrak{M}_n$-modular then there exists $C \in D$ (definition D as in Proposition 3) such that $CH = X$.

__Proof.__ The case $d = n$ is trivial. Suppose now that for $n - 1$ the affirmation has been proved. By induction (see [2]) for n we can suppose that $H \equiv X \mod X_n$. Again by [2] there exists $C_1 \in D$ such that $H_1 = C_1 H \equiv X \mod X_n^3$ and thus for some unimodular $Q \in A_n^t$

$$(6) \qquad Q \equiv \begin{pmatrix} 0 \\ \cdot \\ 1 \\ 0 \\ \cdot \\ 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \cdot \\ 1\end{matrix}}\right\} n \\ \\ \\ \end{matrix} \mod X_n$$

the product

$$(6') \qquad Q^* H_1 = X_n$$

By (6) and 4) as it is well known there exists $C_2 \in GL(t, A_n, X_n)$ with Q as the n-th row. Hence by (6′) the n-th element in the column $H_2 = C_2 H_1$ is X_n and still $H_2 \equiv X \mod X_n$. Eventually applying matrices

$$\begin{pmatrix} U & V \\ 0 & W \end{pmatrix}, \quad U \in GL(d, A_n), \ W \in GL(t - d, A_n)$$

we obtain X. The proof is over.

In [5] it was shown that if L was a free algebra over a field in a nilpotent variety and P retract of L, then P was free and $L = P * B$. The following example shows that this condition is not satisfied in metabelian Lie algebras, though by [3] and [4] they are quite close to nilpotent algebras. Let L be a free metabelian algebra over a ring k with the base e_1, e_2. Define $f: L \longrightarrow L$, $f = (h, \varphi)$ as in § 1 by

$$(7) \qquad h(e_1) = e_1 + Xe_2 - Ye_1, \ h(e_2) = 0, \ \varphi(X) = X, \ \varphi(Y) = 0.$$

Then $f^2 = f$. Suppose that there exists a base $u_1 = h(e_1)$, u_2 in M such that $\ell(u_1) = X$, $\ell(u_2) = Y$ and $h(u_2) = 0$. By Theorem 3

$$u_1 = (1 + XYg)e_1 + Y^2 ge_2 \qquad g \in k[X, Y]$$

Via (7) this is not possible. Hence Imf is not a free factor of L.

§ 4. Homomorphisms of free metabelian A_qA-groups. Let $q \geq 0$ and $q \neq 1$. If C_n is a free abelian group with free generators $X_1,\dots,X_n$ consider a group ring $K = \mathbb{Z}/q\mathbb{Z}\ C_n = \mathbb{Z}/q\mathbb{Z}\,[X_1^{\pm 1},\dots,X_n^{\pm 1}]$ with the augmentation ideal $\mathfrak{m} = (X_1 - 1,\dots,X_n - 1)$. Let M be a free K-module with the base $e_1,\dots,e_n$. Define $\ell : M \longrightarrow \mathfrak{m}$ by $\ell(e_i) = X_i - 1$. Following [1],[2] a free A_qA-group F of rank n is a group of all matrices

$$(8) \qquad \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \quad a \in C_n,\ b \in M,\quad \ell(b) = a - 1.$$

The free generators of F are

$$\begin{pmatrix} X_i & 0 \\ e_i & 1 \end{pmatrix} \quad i = 1,\dots,n.$$

Note that by [1] F' consists of all matrices (8) with $a = 1$, or equally $\ell(b) = 0$.

We are going to show that the results similar to those of § 1, 3 hold for metabelian groups. Let C_1 be a free abelian group with the base $X_1,\dots,X_n$; C_2 with the base $Y_1,\dots,Y_f$; $K_i = \mathbb{Z}/q\mathbb{Z}\ C_i$, M_i, $\mathfrak{m}_i$, ℓ_i, $i = 1,2$, correspond to free A_qA-groups F_1 and F_2. Let $f\colon F_1 \longrightarrow F_2$ be a group homomorphism. As in [1] define $\varphi : K_1 \longrightarrow K_2$ and $h\colon M_1 \longrightarrow M_2$ by

$$(9) \qquad f\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} \varphi(a), 0 \\ h(b), 1 \end{pmatrix}$$

Thus by (9) we define group homomorphism $\varphi : C_1 \longrightarrow C_2$ which in its turn determines ring homomorphism $\varphi : K_1 \longrightarrow K_2$. An easy calculation based on matrix multiplication shows that h is a φ-semilinear homomorphism $h\colon M_1 \longrightarrow M_2$. Note that by (9)

$$(9') \qquad \ell_2(h(b)) = \varphi(a) - 1 = \varphi(\ell_1(b))$$

or equally, the following diagram is commutative

$$(9'') \qquad \begin{array}{ccc} M_1 & \xrightarrow{\ell_1} & \mathfrak{M}_1 \\ h \downarrow & & \downarrow \varphi \\ M_2 & \xrightarrow{\ell_2} & \mathfrak{M}_2 \end{array}$$

Conversely, if $\varphi : C_1 \longrightarrow C_2$ is a group homomorphism, $h:$ $: M_1 \longrightarrow M_2$ is a φ-semilinear morphism and (9'') is commutative, then by (9) the pair (h, φ) determines group homomorphism $f = (h, \varphi): F_1 \longrightarrow F_2$. It is clear that this correspondence is one-to-one and is functorial.

Theorem 5. Let $f: F_1 \longrightarrow F_2$ be an epimorphism of free A_qA-groups, $q \geq 0$, $q \neq 1$, $\operatorname{rk} F_1 = n$, $\operatorname{rk} F_2 = d$. Then there exists a base $z_1, \dots, z_n$ in F_1 such that $f(z_1), \dots, f(z_d)$ is a base for F_2 and $z_{d+1}, \dots, z_n$ generate Kerf as a normal subgroup.

Corollary. Let P be a retract of a free A_qA-group F with a projection $f: F \longrightarrow P$. Then F possesses a base $z_1, \dots$ $\dots, z_n$ as in Theorem 5 and in addition $f(z_i) \equiv z_i \mod \operatorname{Ker} f$, $i = 1, \dots, d$.

The proof follows immediately from freeness of P (see [2]).

Proof of Theorem 5. First we assume that $q = 0$ or q is a prime. If $f: F_1 \longrightarrow F_2$ is onto as in § 3 we can assume that

$$(10) \qquad \varphi(X_i) = \begin{cases} X_i, & i = 1, \dots, d, \\ 1, & i = d+1, \dots, n. \end{cases}$$

Put $J = (X_{d+1} - 1, \dots, X_n - 1) \lhd K_1$. If $A_1 = \mathbb{Z}/q\mathbb{Z}\,[X_1^{\pm 1}, \dots$

$\ldots, X_i^{\pm 1}]$,then by [8] the conditions 1) - 4) in § 3, where X_i stands for $X_i - 1$, are satisfied. Hence, as in the proof of Theorem 4 we can choose in M_1 a new base $u_1, \ldots, u_n$ such that if $f = (h, \varphi)$, then

$$\ell_1(u_i) = X_i - 1, \ i = 1, \ldots, n;$$
$$u_j \in \operatorname{Ker} h, \quad j = d + 1, \ldots, n,$$

and $h(u_1), \ldots, h(u_d)$ is the base for M_2. By (9),(9′),(9″) and (10)

$$z_i = \begin{pmatrix} X_i & 0 \\ u_i & 1 \end{pmatrix}$$

is the necessary base for F_1 (see [1, 2]). Thus in the case $q = 0$ or q prime the theorem is proved.

Suppose now that $q = p^t$, where p is a prime, and $f\colon F_1 \to F_2$ as in the theorem. Let $N_i \lhd F_i$ be a verbal subgroup in F_i corresponding to the subvariety $A_pA \subset A_qA$. Then f induces $f'\colon F_1/N_1 \longrightarrow F_2/N_2$. By the preceding results there exists a base $z_1', \ldots, z_n'$ in F_1/N_2 associated with f'. By [2] there is a base $z_1, \ldots, z_n$ in F_1 such that $z_i \equiv z_i' \bmod N_1$. By the same argument $f(z_1), \ldots, f(z_d)$ is a base for F_2. Thus,

$$f(z_j) = g_j(f(z_1), \ldots, f(z_d)), \ j = d + 1, \ldots, n$$

and hence,

$$z_1, \ldots, z_d, \ z_j\, g_j^{-1}(z_1, \ldots, z_d), \ j = d + 1, \ldots, n$$

is the base we need.

Finally we have to consider the case of arbitrary $q > 2$. Let q have a prime-power factorization $q = \prod q_i$ with prime powers q_i. Note that q_i are coprime for distinct i. Let f,

F_i, C_i, K_i, M_i, $\mathcal{m}_i$, ℓ_i, $i = 1,2$, be as above. Put $s_i = qq_i^{-1}$ and consider a $\mathbb{Z}/q_j\mathbb{Z}\ C_i$-module s_jM_i with epimorphism of $\mathbb{Z}/q_j\mathbb{Z}\ C_i$-modules

$$s_j\ell_i\colon s_jM_i \longrightarrow s_j\,\mathcal{m}_i.$$

As in [2] the group F_{ij} of all matrices

$$\begin{pmatrix} a & 0 \\ s_jb & 1 \end{pmatrix}, \quad a \in C_i,\ b \in M_i,\ s_j(a-1) = s_j\ell_i(b)$$

forms a free $A_{q_j}A$-group with free generators

$$\begin{pmatrix} X_i & 0 \\ s_je_i & 1 \end{pmatrix}, \ i = 1,\dots,n.$$

The epimorphism $f\colon F_1 \longrightarrow F_2$ induces epimorphism $f_j\colon F_{1j} \longrightarrow F_{2j}$ for all j. From a prime power case for every j there is a base $z_{1j},\dots,z_{nj}$ in F_{1j} such that images of the first d of them form a base in F_{2j}, the others generate Ker f_j as a normal subgroup. Moreover, as it follows from the preceding case

$$z_{ij} = \begin{pmatrix} X_i & 0 \\ s_ju_{ij} & 1 \end{pmatrix}, \ i = 1,\dots,n.$$

By [2] there exist free generators $z_1,\dots,z_n$ in F_1 such that

$$z_i = \begin{pmatrix} X_i & 0 \\ u_i & 1 \end{pmatrix}$$

and $s_ju_i = s_ju_{ij}$ for all i, j. The same argument shows that images of $z_1,\dots,z_d$ form a free generating set for F_2. Thus as in prime-power case we can construct the necessary base

$z_1,\dots,z_d$, $z_j g_j^{-1}$, $j = d+1,\dots,n$, where $g_j = g_j(z_1,\dots$
$\dots,z_d)$.

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