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GRAPHS WITH PRESCRIBED MAXIMAL SUBGRAPHS AND CRITICAL CHROMATIC GRAPHS

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Abstract: It is proved that k-chromatic-critical graphs of large order contain large subgraphs of a certain structure. One of these results is that each large k-chromatic-critical graph contains a large odd circuit. A more general result is that if a large 2-connected graph G contains subgraphs of a certain structure of order N but not of order > N then G contains at least two disjoint isomorphic subgraphs not linked by an edge which are "isomorphically" connected to the rest $G - H_1 - H_2$ by edges. A so-called p-reduction is studied for such graphs.

Key words: Critical chromatic graph, subgraph, p-reduction.

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1. Graphs with prescribed maximal subgraphs. We consider undirected finite graphs without loops and multiple edges. If we handle with infinite graphs we say this explicitly. Further definitions are used as in [15]. We say that a path t is a topological edge in a graph G, iff all inner vertices have in G the valency 2 and the two endvertices have a valency ≥ 3. The class K of finite graphs is said to have property E if for each graph $G \in K$ it holds: If t is an arbitrary topological edge only the endvertices of which are contained in G then $G + t$ contains a subgraph $G' \in K$ with $t \subseteq G'$.

Now we will present a few of such classes. A graph G is said to be contractible to a graph G' iff there exists a homomorphism φ from G onto G' with the properties:

1) For each vertex X' of G' it holds: The spanning subgraph of the vertex set $\varphi^{-1}(X')$ in G is a tree.

2) For each pair $\{X', Y'\}$ of different vertices of G' it holds:

a) $\varphi^{-1}(X')$ and $\varphi^{-1}(Y')$ are joined by at most one edge.

b) X' and Y' are joined by an edge in G' iff $\varphi^{-1}(X')$ and $\varphi^{-1}(Y')$ are joined by an edge.

That means G' can be obtained from G by consecutive contractions of edges not contained in triangles.

A prismgraph consists of two circuits, which have at most one common vertex and which are united by three vertex-disjoint paths; if the two circuits have a common vertex then one path has length 0.

Theorem 1: The class W of all paths , the class C of all circuits, the class O of all odd circuits, the class P of all prismgraphs, the class $\langle r,S \rangle$ of all 2-connected graphs contractible to a complete r-graph ($v \geq 4$) and the class $V(s_1, s_2, s_3, s_4, \ldots, s_p)$ have property E.

Each graph of the latter class can be formed as follows: We start with a set $\{H_j\}$ of $s_1 + s_2 + s_3 + s_4 + \ldots + s_p$ pairwise disjoint graphs; s_1 of which are isolated vertices, $s_2, s_3, s_4, \ldots, s_p$ are graphs of C, P, $\langle 4,S \rangle, \ldots, \langle p,S \rangle$, respectively. Then we consecutively link two components by a topological edge until we have obtained a connected graph. $V(s_1)$

is the set of all trees with at most s_1 endvertices. $V(s_1,s_2)$ is the set of all Husimi trees with at most s_1 vertices of valency 1 and exactly s_2 circuits.

<u>Proof of Theorem 1</u>: Let $G \in W \cup C \cup O \cup P \cup \ldots$ and let V_1, V_2 be the two different endvertices of a topological edge w with $G \cap w = \{V_1, V_2\}$.

a) Obviously, W and C have E.

b) Let $G \in O$. The two circuit arcs of G between V_1 and V_2 have different parity. Let t denote the one which has the same parity as w. Then $G + w - t \in O$ with $w \subseteq G + w - t$.

c) For the class P the proof can easily be obtained by distinguishing some cases.

d) Let $G \in \langle r,S \rangle$ and let G' denote a complete r-graph. We have only to consider two cases:

1) There exists one vertex X' of G' and a homomorphism φ of G onto G' such that $V_1, V_2 \in \varphi^{-1}(X')$. The spanned subgraph U of $\varphi^{-1}(X')$ is a tree, therefore $U + w$ contains exactly one circuit $\overline{C}$.

Let e be an edge of $\overline{C}$ not contained in w. We delete in $G + w$ the topological edge t of $G + w$ containing e and we have a new tree $U + w - t$. It is easily to be seen that with $\varphi(U + w - t) =_{def} X'$ we have $G + w - t \in \langle r,S \rangle$.

2) There exist two different adjacent vertices X' and Y' of G' and a homomorphism φ of G onto G' such that $V_1 \in \varphi^{-1}(X')$ and $V_2 \in \varphi^{-1}(Y')$. Then there is in G an edge e connecting $\varphi^{-1}(X')$ and $\varphi^{-1}(Y')$. Let t denote the topological edge of $G + w$ containing e. Now it can easily be seen that $w \subseteq G + w - t$ and $G + w - t$ is an element of $\langle r,S \rangle$. Thus

the proof of d) is complete.

e) Let $G \in V(s_1,\dots,s_p)$. If $w \cap H_j = \{V_1, V_2\}$ then the validity of the Theorem can easily be derived from a) ... d). If for all j it holds $\| w \cap H_j \| \leq 1$ then G + w has a circuit containing w and a topological edge t with $t \cap w \subseteq \{V_1, V_2\}$ and for all j it holds $\| t \cap H_j \| \leq 1$. Then $w \subseteq G + w - t \in V(s_1,\dots,s_p)$.

Q.e.d.

Remark: The class of all circuits containing a certain vertex X also has property E but the class of all circuits containing two certain vertices X and Y has not property E.

Theorem 2: a) Let K denote a class of finite graphs with property E and let N be a positive integer. Let G be a 2-connected (finite or infinite) graph which contains a graph $H \in K$ of order N but which does not contain an element of K of order $> N$. Then the length l of a maximal circuit L of G is $l \leq N^2$.

b) If K = O then $\ell \leq 2(N - 1)$.

In a) the bound is not best possible. That in b) the bound is best possible is shown by the graph which consists of two vertices of valency 3 which are linked by an edge and by two topological edges of length $N - \ell$(N odd).

By Theorem 2b in each 2-connected nonbipartite graph G it holds $\ell^* \leq \ell \leq 2(\ell^* - 1)$ where ℓ, ℓ^* denote the maximal circuit length and the maximal odd circuit length, respectively (provided ℓ^* exists).

A similar assertion for the maximal even circuit length does not hold. G.A. Dirac proved in [2] that in 2-connected finite graphs it holds that $\ell - 1 \leq \bar{\ell} \leq \ell^2$ where $\bar{\ell}$ denotes the maximal path length of G. It can be shown that $\ell - 1 \leq \bar{\ell} \leq \ell^2/4$ (see G.A. Dirac [2], see also [15] and [13] . In the following $\lambda(H)$ and $o(H)$ denote the number of edges or vertices of H, respectively.

<u>Proof of Theorem 2</u>: a) We distinguish two cases.

1) Let $\|L \cap H\| \leq 1$. Then there exist two disjoint paths w_1, w_2 connecting L and H (possibly $\lambda(w_i) = 0$). Let $X_i =_{def} w_i \cap H$ and $Y_i =_{def} w_i \cap L$. Let L_1, L_2 denote the two circuit arcs of L between Y_1 and Y_2. $p_i = w_1 + L_i + w_2$ $(i = 1,2)$ are two paths with $p_i \cap H = \{X_1, X_2\}$.

By Theorem 1 it follows that $H + p_i$ contains a subgraph H_i with $p_i \subseteq H_i \in K$. Therefore

$$\lambda(w_1) + \lambda(L_i) + \lambda(w_2) = \lambda(p_i) \leq o(H_i) \leq o(H) = N.$$

Hence $\lambda(L_i) \leq N - 1$ and $\ell = \lambda(L) \leq 2(N - 1)$.

2) Let $\|L \cap H\| \geq 2$. Then L can be split up in arcs $L_1, L_2, \ldots$ $\ldots, L_q$ such that for each i it holds:

Either $L_i \cap H = L_i$ or $L_i \cap H = \{P_1^i, P_2^i\}$ where P_1^i, P_2^i are the two endvertices of L_i. In both cases we have $\lambda(L_i) \leq o(H) = N$ (in the case $L_i \cap H = \{P_1^i, P_2^i\}$ see 1)). Because $q \leq N$ it follows $\lambda(L) \leq N^2$.

b) Let $H \in O$. If L is and odd circuit $\lambda(L) = o(H) = N$. Now let L be an even circuit.

If $\|H \cap L\| \leq 1$ the assertion b) of the Theorem follows from the proof a1).

Now let $\|H \cap L\| \geq 2$. Let P and Q be two arbitrary vertices of $H \cap L$. Because $H \in O$ and L is an even circuit the parity of one of the two circuit arcs of H between P and Q is different from the parity of the two circuit arcs of L between P and Q. From this it can easily be derived that there exists a circuit arc $\overline{H}$ of H with $\|\overline{H} \cap L\| = 2$ and the parity of $\overline{H}$ is different from the parity of the two circuit arcs L_1, L_2 of L connecting the two vertices of $\overline{H} \cap L$ in L. Hence $L_i + \overline{H} \in O$ and

$$\lambda(L) + 2\,\lambda(\overline{H}) = \lambda(L_1 + \overline{H}) + \lambda(L_2 + \overline{H}) \leq 2\,N.$$

With $\lambda(\overline{H}) \geq 1$ the Theorem 2 is proved.

Let p be a positive integer. Now we will define the so called p-reduction of finite and infinite graphs described in [15]. Two finite subgraphs U_1, U_2 of a finite or infinite graph G are called independent, if they do not have a common vertex and if they are not connected by an edge. Two finite subgraphs U_1, U_2 of G are called equivalent, if $U_1 \equiv U_2$ or if U_1 and U_2 are independent and there exists an isomorphism of the grap $G - U_1$ onto the graph $G - U_2$ such that all vertices of $G - U_1 - U_2$ are fixed.

Let M be a class of pairwise independent finite subgraphs of G, then the above formulated so called equivalence is an equivalence relation in M. Therefore M is divided in equivalence classes. From each equivalence class with more than p elements we delete in G so many elements of this equivalence class that in G only p elements remain. We call this deletion an "elementary p-reduction".

A sequence of a finite number of elementary p-reductions is called a p-reduction. If the obtained graph is denoted by G' then we write $G \succ G'$.

Let K be a class of finite graphs with property E. Let N be an integer.

Z(K,N) denotes the class of all 2-connected finite and infinite graphs which contain an element of K of order N but which do not contain an element of K of order $> N$. For this class we can prove a finiteness condition in the following sense:

Each large graph $G \in Z(K,N)$ contains $p + 1$ equivalent subgraphs; that means to each positive integer p there exists a positive integer $n(p,K,N)$ such that every $G \in Z(K,N)$ of order $\geq n(p,K,N)$ contains $p + 1$ equivalent subgraphs. We define

$$\alpha(C,N) = \left[\frac{N}{2}\right] + 1 \text{ and } \beta(C,N) = \left[\frac{N}{2}\right] - 1,$$

$$\alpha(O,N) = N \quad \text{and} \quad \beta(O,N) = N - 2,$$

$$\alpha(K,N) = \beta(K,N) = [N^2/2] + 1, \text{ if } K \neq C,\ K \neq O.$$

<u>Theorem 3</u>: Let p, N be integers with $N \geq 3$ and $p \geq \alpha(K,N)$. Then

a) To each $G \in Z(K,N)$ there exists a p-irreducible graph G' with $G \succ G'$. The graph G' can be obtained from G by a sequence of at most $\beta(K,N)$ elementary p-reductions.

b) Every p-irreducible graph G' with $G' \prec G$ is up to isomorphism uniquely determined, is finite and it is also $G' \in Z(K,N)$.

c) $Z(K,N)$ only contains a finite number of unisomorphi p-irreducible graphs.

In Theorem 3a) for some graphs of $Z(C,N)$ and $Z(O,N)$ we really need $\beta(C,N) = \left[\frac{N}{2}\right] - 1$ or $\beta(O,N) = N - 2$ elementary p-reductions, respectively, to obtain the p-irreducible graph. For $Z(C,N)$ we have shown this in [15]. For $Z(O,N)$ this is proved by the following graph:

We consider a tree T with a root X such that the distance between X and each endvertex is $N - 2$, that each inner vertex has a valency $\geq p + 1$ and that every two inner vertices have different valencies. To this tree we add a new vertex Y and link it with X and the endvertices of T by edges.

Let $G \circ X$ denote the graph obtained from a graph G by adding a new vertex X and linking X to each vertex of G by an edge. Let $\overline{Z}(W,N)$ be the class of all connected graphs containing a path of length N but no path of length $> N$. Then $G \in \overline{Z}(W,N)$ iff $G \circ X \in Z(C,N + 2)$. Therefore it yields the

<u>Remark</u>: The Theorem 3 is also valid for $\overline{Z}(W,N)$ with $\overline{\alpha}(W,N) = [\frac{N}{2}] + 2$ and $\overline{\beta}(W,N) = [\frac{N}{2}]$.

<u>Proof of Theorem 3</u>: In the case $K = C$ the Theorem was proved in [15], it is not proved here again. If $K \neq C$ then from Theorem 2 it follows:

$$Z(O,N) \subseteq \bigcup_{i=N}^{2(N-1)} Z(C,i) \text{ and}$$

$$Z(K,N) \subseteq \bigcup_{i=3}^{N^2} Z(C,i), \text{ if } K \neq O.$$

By applying the result already known for $Z(C,i)$ we obtain the Theorem also in the case that $K \neq C$. It remains only to show that if $G \in Z(K,N)$ and $G \vdash G'$ then G' contains a subgraph of K of order N. But this can easily be done by taking the following into consideration: If $H \in K$, $H \subseteq G$ and $o(H) = N$, then each p-reduction can be chosen such that no vertex of H is deleted (notice that $p \geq N$ if $K \neq C$ and $p \geq N/2 + 1$ if $K = C$). Q.e.d.

2. <u>Critical chromatic graphs</u>. In 2. we only consider finite graphs. The chromatic number of a graph G is denoted by $\chi(G)$. A graph is called k-critical, iff its chromatic number is k and by the deletion of an arbitrary edge the resulting graph has the chromatic number k - 1. In this paper only k-critical graphs with $k \geq 3$, are considered.

<u>Lemma</u>: Let G, G' be graphs with $G \vdash G'$. Then $\chi(G') = \chi(G)$.

<u>Proof</u>: It suffices to show that if U_1 and U_2 are two equivalent subgraphs of G then $\chi(G - U_2) = \chi(G)$. But this can be seen by the fact that each suitable colouring of G - - U_2 can be extended to a suitable colouring of G by giving the same colour to the vertices X and $\varphi(X)$ for each $X \in U_1$ whereby φ denotes an isomorphism of $G - U_2$ onto $G - U_1$ with fixed $G - U_1 - U_2$. Q.e.d.

It is well known that each critical graph is 2-connected. Z(K,N,k) denotes the set of all k-critical graphs $G \in Z(K,N)$.

<u>Theorem 4</u>: Let K be a class of graphs with property E. Then Z(K,N,k) contains only a finite number of nonisomorphic graphs.

<u>Proof</u>: The Lemma shows that all graphs $G \in Z(K,N,k)$ are 1-irreducible and also p-irreducible. Because Z(K,N) only contains a finite number of p-irreducible graphs (Theorem 3) the truth of Theorem 4 follows from $Z(K,N,k) \subseteq Z(K,N)$. Q.e.d.

Theorem 4 states that each k-critical graph of large order which has an element of K as a subgraph contains also a large graph of K. If F(K,n,k) is the largest integer such that every k-critical graph of order n which has an element of K as a subgraph, contains a subgraph of K of order $\geq F(K,n,k)$ then

$$(1) \qquad \lim_{n \to \infty} F(K,n,k) = +\infty .$$

Obviously, for $k \geq 3$ each k-critical graph contains a subgraph $H' \in C$ and a subgraph $H'' \in O$. Thus each large k-critical graph contains a large circuit and also a large odd circuit. The first assertion was proved by J.B. Kelly and L.M. Kelly [10] in 1954, the second assertion gives an answer of case $\varkappa = 3$ of the question posed by J. Nešetřil and V. Rödl at the International Colloquium on Finite and Infinite Sets held in 1973 in Keszthely in Hungary(oral communication):

<u>Problem</u>: Let $\varkappa$,k,N be arbitrary positive integers with $\varkappa < k$. Does there exist a positive integer n such that each k-critical graph G with at least n vertices contains a $\varkappa$-critical subgraph G' with at least N vertices?

The order of the magnitude of $F(C,n,k)$ was investigated by J.B. Kelly and L.M. Kelly [10], G.A. Dirac [3] and R.C. Read [12]. T. Gallai [8] has obtained a sharpening of these results by showing that for an infinite set of different positive integers n there exist k-critical graphs of order n of maximal circuit length $\leq c_k \log n$, where c_k is an appropriate constant. From Theorem 2b) it follows that $F(C,n,k)$ and $F(O,n,k)$ have the same magnitude.
It also yields that the result "each large k-critical graph contains a large odd circuit" can be derived from the result of Kelly/Kelly "each large k-critical graph contains a large circuit" by means of Theorem 2b.

Before discussing other classes K we define: If $r \geq 3$, then a topological complete r-graph consists of r branching vertices and of $\binom{r}{2}$ topological edges such that every two branching vertices are linked by exactly one topological edge. $\langle r,U \rangle$ denotes the class of all topological complete r-graph .

G.A. Dirac [1] has proved that each 4-critical graph contains a $\langle 4,U \rangle$. B. Zeidl [16] has shown that for $k \geq 4$ each k-critical graph has a $\langle 4,U \rangle$, containing a circuit of odd length.

In [4] G.A. Dirac has proved that each circuit of a 4-critical graph is contained in a $\langle 4,U \rangle$. If we apply this result to the largest circuits and to the largest odd circuits, then we obtain from (1) with respect to K = C and K = O: For $k \geq 4$ each large k-critical graph has a large $\langle 4,U \rangle$ and also a large $\langle 4,U \rangle$ containing a circuit of odd length, respectively.

In this paper I proved the first statement again (see (1)) but I cannot reprove the second statement with the aid of Theorem 3 because the class of all graphs of $\langle 4,U \rangle$ containing an odd circuit has not property E.

Because each k-critical graph has no vertex of valency $\leq k - 2$, every k-critical graph of order n has $\geq \frac{1}{2}(k - 1)$ n edges. This lower bound was improved by T. Gallai [8]and G.A. Dirac [6]. For $k \geq 6$ each k-critical graph contains at lest $\frac{5}{2}n$ edges. A result of G.A. Dirac [5] says that each simple graph of order $n \geq 5$ with at least $\frac{5}{2}n - 3$ edges contains a graph obtained from a graph of the class $\langle 5,U \rangle$ by deleting one and only one topological edge. Because this graph has a special prismgraph it follows: For $k \geq 6$ each k-critical graph contains a prismgraph and with (1) each large k-critical graph contains a large prismgraph.

K. Wagner [14] (H.A. Jung [9]) has proved: For every positive integer r there exists an integer k_r (an integer k_r') such that for all positive integers $k \geq k_r$ ($k \geq k_r'$) each k-critical graph contains a $\langle r,S \rangle$ (a $\langle r,U \rangle$) - also see W. Mader [11]. By (1) from this it follows: For all $k \geq k_r$ each large k-critical graph contains a large $\langle r,S \rangle$.
But by our methods it cannot be shown that for all $k \geq k_r'$ each large k-critical graph contains a large $\langle r,U \rangle$ because $\langle r,U \rangle$ has not property E. We do also not know whether this assertion is true.

By definition we have

$$\bigcup_{k \geq 2} Z(K,N,k) \subseteq Z(K,N).$$

Because by Theorem 3c the number of nonisomorphic graphs of Z(K,N) is finite we have that there exists a positive integer k(K,N) such that $Z(K,N,k) = \emptyset$ for all $k \geq k(K,N)$. By a result of P. Erdös and H. Hajnal we can take $k(0,N) = N + 2$ because they showed in [7]: Every graph which does not contain circuits of lengths $2j + 1$ for all $j \geq i$ is suitable colourable by 2i colours.

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